The Local Dyadic Conjecture

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Abstract. A rational with denominator a power of two is said to be dyadic. A long standing conjecture of Paul Seymour predicts that for an ideal clutter, the dual of the set covering problem always admits an optimal solution where all variables take dyadic values. We present a local version of this conjecture, and motivate it by proving an important special case.

Keywords: Linear Programming, dyadic rationals, binary clutters, polyhedra

1 Introduction.

1.1 The dyadic conjecture.

A family of sets $\mathcal{C}$ over finite ground set $E(\mathcal{C})$ is a *clutter* if for every $S, S' \in \mathcal{C}$ where $S \subseteq S'$ we have $S = S'$. A clutter $\mathcal{C}$ is *trivial* if $\mathcal{C} = \emptyset$ or $\mathcal{C} = \{\emptyset\}$. Throughout this paper we shall assume that all clutters are non-trivial. Given a clutter $\mathcal{C}$, $M(\mathcal{C})$ denotes the $0, 1$ matrix where the rows correspond to the characteristic vectors of members of $\mathcal{C}$. [1] Consider a clutter $\mathcal{C}$, and weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$. We consider the following primal-dual pair,

$$\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}, \tag{1}$$

$$\max\{\mathbf{1}^\top z : M(\mathcal{C})^\top y \leq w, y \geq \mathbf{0}\}. \tag{2}$$

Given a clutter $\mathcal{C}$ we associate the polyhedron,

$$P(\mathcal{C}) := \{x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}.$$

The clutter $\mathcal{C}$ is *ideal* if $P(\mathcal{C})$ is integral, i.e. every vertex (also known as extreme point) of $P(\mathcal{C})$ is in the integer lattice. In 1975, Paul Seymour (see [19] 79.3e) proposed the following conjecture,

Conjecture 1. Let $\mathcal{C}$ be an ideal clutter. Then for every $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ the linear program (2) has an optimal solution that is dyadic.

[1] $M(\mathcal{C})$ is only defined up to permutations.

Here, a rational number $\frac{r}{s}$ where $r, s \in \mathbb{Z}$ is *dyadic* if $s = 2^q$ for some $q \geq 0$ where $q \in \mathbb{Z}$. A vector is *dyadic* if every entry is dyadic.

Dyadic numbers are important for numerical computations because they have a finite binary representation, and therefore they can be represented exactly on a computer in floating-point arithmetic. Checking whether a linear program admits a dyadic optimal solution can be done in polynomial time [2]. For an implementation see [6]. For the set packing problems, integrality of the polytope, $\{x \geq \mathbf{0} : M(\mathcal{C})x \leq \mathbf{1}, x \geq \mathbf{0}\}$ implies the existence of an integral solution to the dual problem $\min\{\mathbf{1}^\top y : M(\mathcal{C})^\top y \geq w, y \geq \mathbf{0}\}$ for every $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ [5]. This affirms a one-to-one correspondence between perfect matrices and perfect graphs. Conjecture 1 would establish a weak counterpart for set covering problems.

Conjecture 1 is known to hold for a number of special cases, for instance, (a) ideal clutters of odd cycles of graphs [9], (b) clutters of T-cuts of grafts [22], (c) clutters of T-joins of grafts [1], (d) clutters of dicuts of a directed graph [14], (e) clutters of dijoins of a directed graph [11] see also [4]. For (a), (b) (2) always have an $\frac{1}{2}$-integral solution. For (d) (2) always has an integral solution.

1.2 A local strengthening of the Dyadic Conjecture.

Consider a pointed polyhedron $P \subseteq \mathbb{R}^n$. We say that a pair of vertices v^1, v^2 of P are *adjacent* if both v^1, v^2 are contained in a single face of dimension 1 of P. We say that a vertex v of a polyhedron P is an *integral corner*, if v and all vertices adjacent to v are in $\mathbb{Z}^n$. Consider now a clutter $\mathcal{C}$ and weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$. Observe that $\min\{w^\top x : x \in P(\mathcal{C})\}$ always has an optimal solution and that the set of optimal solutions is a face F of $P(\mathcal{C})$. We say that F is the *w-optimal face* of $P(\mathcal{C})$. A set $B \subseteq E(\mathcal{C})$ is a *cover* of $\mathcal{C}$ if $B \cap S \neq \emptyset$ for every $S \in \mathcal{C}$. A cover is w-minimum if it minimizes $w(B) := \sum_{j \in B} w_j$ among all covers. We say that w are *minimal* weights of $\mathcal{C}$ if $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ and every $j \in \mathcal{C}$ is contained in a w-minimum cover.

We predict the following.

Conjecture 2. Let $\mathcal{C}$ be a clutter and let w be minimal weights of $\mathcal{C}$. If the w-optimal face of $P(\mathcal{C})$ contains an integral corner then (2) has an optimal solution that is dyadic.

Observe that for an integral polyhedron, every vertex is an integral corner. In particular, if $\mathcal{C}$ is ideal then every vertex of $P(\mathcal{C})$ is an integral corner. Furthermore, it is an easy exercise to show that it suffices to prove Conjecture 1 for minimal weights only. Hence, Conjecture 2 implies Conjecture 1. However, the former appears strictly stronger than the latter.

The condition that the optimal face contain an integral corner cannot be omitted. Consider for instance the case where $\mathcal{C}$ are the lines of the Fano Matroid, i.e.

$$\mathcal{C} = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}\}.$$

For minimal weights $w = \mathbf{1}$ the w-optimal face contains a unique point $\frac{1}{3}\mathbf{1}$ and (2) has a unique dual solution $\frac{1}{3}\mathbf{1}$ as well.

It is also not possible to replace in Conjecture 2 the condition that "*the w-optimal face contains an integral corner*" by "*the w-optimal face contains an integral point*". To see this consider the clutter $\mathcal{C}$ with,

$$M(\mathcal{C}) = \begin{pmatrix} 0&1&1&1&1&0&0&0 \\ 1&0&1&1&0&1&0&0 \\ 1&1&0&1&0&0&1&0 \\ 1&1&1&0&0&0&0&1 \\ 0&0&0&0&1&1&1&1 \end{pmatrix}.$$

Let $w = \mathbf{1}$ and let F denote the w-optimal face. It can be readily checked that (2) has a unique optimal solution $y = (\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3})^\top$. The vertices of F are $v^1 = (1,0,0,0,1,0,0,0)$, $v^2 = (0,1,0,0,0,1,0,0,0)$, $v^3 = (0,0,1,0,0,0,1,0)$, $v^4 = (0,0,0,1,0,0,0,1)$. We claim that none of v^i, $i \in [4]$ is an integral corner. Because of symmetry it suffices to consider v^1, but v^1 is adjacent to $(\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,1)^\top$.

1.3 A special case of Conjecture 2.

The set of all inclusion-wise minimal covers of a clutter $\mathcal{C}$ forms another clutter called the *blocker* of $\mathcal{C}$. We denote by $b(\mathcal{C})$ the blocker of $\mathcal{C}$. The blocking operation is idempotent, i.e., $b(b(\mathcal{C})) = \mathcal{C}$ [13,8]. A clutter $\mathcal{C}$ is said to be *binary* if for every $S \in \mathcal{C}$ and every $B \in b(\mathcal{C})$ we have $|S \cap B|$ odd. Cases (a)-(c) of Conjecture 1 are all instances of binary clutters. The restriction of Conjecture 1 to binary clutters remains open and is therefore of particular interest.

A main corollary of this paper is the following special case of Conjecture 2.

Theorem 1. *Let $\mathcal{C}$ be a binary clutter and let w be minimal weights of $\mathcal{C}$. If the w-optimal face of $P(\mathcal{C})$ contains a (k,ℓ)-degenerate integral corner where $k(\ell+1) \le 6$ then* (2) *has an optimal solution that is dyadic.*

A vertex v of a full-dimensional polyhedron $P = \{x : Ax \le b\} \subseteq \mathbb{R}^n$ is (k,ℓ)-*degenerate* if (i) v is contained in $n+k$ faces of dimension 1 of P, and (ii) there are $n+\ell$ tight constraints of $Ax \le b$ for v. Observe that a $(0,0)$-degenerate vertex is non-degenerate. For $\mathcal{Q}_6$, the clutter with groundset the edges of K_4 and members the triangles of K_4, every vertex of $P(\mathcal{Q}_6)$ is (k,ℓ)-degenerate, for $k,\ell \le 2$. For the case where the integral corner is non-degenerate, it can be readily shown that we in fact get an integral solution for (2) as the left-hand-side of the tight constraints must be unimodular [20].

1.4 Organization of the paper.

Section 2 presents a general result, Theorem 2, and explains how Theorem 1 follows as a corollary. Section 3 proves a result about binary clutters that might be of independent interest. In Section 4 we leverage these result and a general tool on dyadic linear programs to prove Theorem 2. Proofs of

- the general tool and
- the fact that (2) has a half-integer solution, when there are no more than 6 active dual variables and the clutter is binary,

are in the full version of this article.

2 The general result.

Before we can proceed, we require a number of definitions. Consider a full-dimensional polyhedron $P = \{x : Ax \leq b\}$ with a vertex v and a proper face F that contains v. Let $\bar{A}x \leq \bar{b}$ denote the implicit equalities for F. A set $\mathcal{L}$ of faces of dimension 1 containing v is *(F, v)-saturating* if no constraint of $\bar{A}x \leq \bar{b}$ is tight for every $L \in \mathcal{L}$. Observe that this is a property of the polyhedron, not of the system of linear constraints.

If F, F' are faces of P where $F' \supset F$ then F' is a *superface* of F. If in addition $\dim(F') = \dim(F) + 1$ then F' is an *immediate superface* of F. Denote by $\mathcal{L}'$ the set of faces of dimension 1 containing v and that contain a point distinct from v in some immediate superface of F. Then F is *v-compatible* if $\mathcal{L}'$ intersects every (F, v)-saturating set. We illustrate these definitions in Figure 1. For both

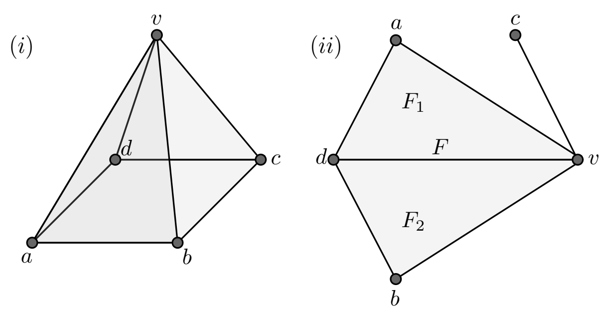

Fig. 1. Illustrating definitions.

(i) and (ii) denote by L_a, L_b, L_c, L_d be the line segment between v and a, b, c, d respectively. Consider (i). The point v is $(1, 1)$-degenerate. Pick $F = \{v\}$. Then the sets $\{L_a, L_c\}$ and $\{L_b, L_d\}$ are both (F, v)-saturating and every other (F, v)-saturating set is a superset of these sets. The immediate superfaces of F are

L_a, L_b, L_c and L_d. The set $\mathcal{L}'$ defined above is $\{L_a, L_b, L_c, L_d\}$. In particular, F is v-compatible. Indeed, whenever F consists of a single point v, then it is always v-compatible. Consider (ii). Pick $F = L_d$. Then the sets $\{L_a, L_b\}$ and $\{L_c\}$ are both (F, v)-saturating. The immediate superfaces of F are the triangles F_1 and F_2. The set $\mathcal{L}'$ defined above is $\{L_a, L_b\}$. Since $\{L_c\} \cap \mathcal{L}' = \emptyset$, F is not v-compatible.

Suppose P is full dimensional and determined by an irreducible system $Ax \leq b$. Given a proper face F of P, we denote by $\sigma(F, P)$ the number of tight constraints of $Ax \leq b$ for F. Note, that $\sigma(F, Q)$ is independent of the choice of $Ax \leq b$. When clear from context, we write $\sigma(F)$ for $\sigma(F, P)$.

We can now state the main result of the paper.

Theorem 2. *Let $\mathcal{C}$ be a binary clutter and let w be minimal weights of $\mathcal{C}$. Denote by $\bar{F}$ the w-optimal face of $P(\mathcal{C})$ and let v be an integral corner of $\bar{F}$. If every face $F \supseteq \bar{F}$ for which $\sigma(F) \geq 7$ is v-compatible then* (2) *has an optimal solution that is dyadic.*

We will show this implies Theorem 1. We first require.

Proposition 1. *Let P be a pointed polyhedron of dimension n and let F be a face of P of dimension d. Then there exists at least $n - d$ immediate superfaces containing F.*

Proof. We first consider the case when P is a polytope. Denote by $L(P)$ the face poset of P. It is a graded lattice with rank function $r(G) = \dim(G) + 1$ for faces G. The interval $[F, P]$ of $L(P)$ is the face lattice of a polytope P' of dimension $r(P) - r(F) - 1 = n - d - 1$ (see [23] Theorem 2.7). $F = \emptyset$ in P' and every immediate superface of F is a vertex of P'. Since $\dim(P') = n - d - 1$, P' must have at least $n - d - 1 + 1$ vertices (equality holds for simplexes). Hence, F is contained in at least $n - d$ immediate superfaces, as required.

Consider now the general case where P need not be a polytope. Pick $M \in \mathbb{Z}_+$ large enough such that every vertex of P is contained in the interior of the box $B := \{x : -M\mathbf{1} \leq x \leq M\mathbf{1}\}$. Describe P by an irreducible set of inequalities $Ax \leq b$ where A has index rows $[m]$. If G is a face of P, then there exists $I_G \subseteq [m]$ such that $G = P \cap \{x : \mathrm{row}_i(A)x = b_i, i \in I_G\}$. Moreover, if G is a face of $P \cap B$ where $G \cap F \neq \emptyset$ for a face F of P, then none of the constraints of B are tight for F as F must contain a vertex of P. Hence, there exists a bijection between faces $G \supseteq F$ of P and faces $G' \supseteq F$ of $P \cap B$. Therefore, we can count the number of superfaces of F in $P \cap B$ rather than P and use the aforementioned result for polytopes. $\square$

Next we tie degeneracy to the number of active dual variables.

Proposition 2. *Let $P \subset \mathbb{R}^n$ be a full dimensional polyhedron. Let v be a (k, ℓ)-degenerate vertex of P where $k(\ell + 1) \leq \alpha$ for some $\alpha \in \mathbb{Z}_+$. Let $F \ni v$ be a face of P with $\sigma(F, P) > \alpha$. Then F is v-compatible.*

Proof. Let $Ax \leq b$ be an irreducible representation of A and let $\bar{A}x \leq \bar{b}$ denote the tight constraints for v. Let $\mathcal{L}$ be the set of all faces of dimension 1 of P containing v.

Claim 1. Let $A'x \leq b'$ be a subset of δ constraints of $\bar{A}x \leq \bar{b}$ and let $L \in \mathcal{L}$. Then $\geq \delta - \ell - 1$ of the constraints $A'x \leq b'$ are tight for L.

Proof of Claim. Since L is a face of dimension 1 and $P \subseteq \mathbb{R}^n$, there are $\geq n-1$ constrains of $\bar{A}x \leq \bar{b}$ that are tight for L. As v is (k,ℓ)-degenerate, $\bar{A}x \leq \bar{b}$ has at most $n+\ell$ constraints. Hence, at most $n+\ell-(n-1) = \ell+1$ of the constraints of $\bar{A}x \leq \bar{b}$ are strict for L. As $A'x \leq b'$ is a subsystem of $\bar{A}x \leq \bar{b}$ the number of tight constraints for L is $\geq \delta - (\ell+1)$. $\diamond$

Let $F_1, \ldots, F_p$ denote the immediate superfaces of F and let $\mathcal{F} = F \cup_{i \in [p]} F_i$.

Claim 2. At least n of the faces in $\mathcal{L}$ intersect $\mathcal{F} \setminus \{v\}$.

Proof of Claim. Let $d = \dim(F)$. Then $p \geq n - d$, by Proposition 1. Since F has dimension d, at least d of the faces in $\mathcal{L}$ intersect $F \setminus \{v\}$. For each $i \in [p]$, at least one face $L_i \in \mathcal{L}$ intersect $F_i \setminus \{v\}$. Pick distinct $i, j \in [p]$. As F_i, F_j are immediate superfaces and $F_i \neq F_i$, we have $L_i \neq L_j$. Hence, we have $d+p \geq d+(n-d) \geq n$ distinct faces of $\mathcal{L}$ that intersect $\mathcal{F} \setminus \{v\}$. $\diamond$

Consider now a face F and let $A'x \leq b'$ be the corresponding subset of tight constraints of $\bar{A}x \leq \bar{b}$. We may assume that there exists a saturating set $\mathcal{L}' \subseteq \mathcal{L}$ where none of the faces in $\mathcal{L}'$ intersect $\mathcal{F} \setminus \{v\}$, for otherwise F is v-compatible. Since v is (k,ℓ)-degenerate, $|\mathcal{L}| \leq n+k$. By Claim 2, n of the faces in $\mathcal{L}$ intersect $\mathcal{F} \setminus \{v\}$. It follows that $|\mathcal{L}'| \leq n+k-n \leq k$. By Claim 1, at most $k(\ell+1)$ constraints of $A'x \leq b'$ are non-tight for $\mathcal{L}'$. Since $\mathcal{L}'$ is saturating, $k(\ell+1) \geq \sigma(F,v)$. However, by hypothesis, $\sigma(F,v) > \alpha \geq k(\ell+1)$, a contradiction. $\square$

Proposition 3. *Theorem 2 implies Theorem 1.*

Proof. Let $\alpha = 6$. By hypothesis of Theorem 1, v is (k,ℓ)-degenerated for $k(\ell+1) \leq \alpha$. It follows from Proposition 2 that every face $F \supseteq \bar{F}$ for which $\sigma(F) \geq 7$ is v-compatible. However, then Theorem 2 implies that (2) has an optimal solution that is dyadic as required. $\square$

In the remainder of the paper we shall prove Theorem 2.

3 A lemma on binary clutters.

Here is the key result of the section,

Proposition 4. *Let $\mathcal{C}$ be a binary clutter and let $S \in \mathcal{C}$. Suppose that there exists $B \in b(\mathcal{C})$ where $|S \cap B| \geq 3$. Then there exists $B' \in b(\mathcal{C})$ for which $|S \cap B'| = 3$ and $S \cap B' \subseteq S \cap B$.*

Before we proceed to the proof, observe that the binary assumption cannot be omitted, even if we replace 3 with any constant bound. Namely, we have,

Remark 1. For every $n \in \mathbb{Z}_+$ there exists a clutter $\mathcal{C}$ with $|E(\mathcal{C})| = 2n$ and $S \in \mathcal{C}$, such that

- there exists $B \in b(\mathcal{C})$ with $|S \cap B| = n$, and
- for every $B \in b(\mathcal{C})$, $|S \cap B| \in \{1, n\}$.

Proof. Let $\mathcal{B}$ be the clutter with groundset $\{1, \dots, n\} \cup \{1', \dots, n'\}$ and members $\{1, \dots, n\}$ and for all $i \in [n]$ members $\{i, i'\}$. Let $\mathcal{C} = b(\mathcal{B})$. Then $S = \{1, \dots, n\} \in \mathcal{C}$ and the result follows. ☐

If we are allowed to pick the set $S \in \mathcal{C}$ then the condition that $\mathcal{C}$ be binary is not needed.

Proposition 5. *Let $\mathcal{C}$ be a clutter. Suppose that some $S \in \mathcal{C}$ and some $B \in b(\mathcal{C})$ satisfy $|S \cap B| > 1$. Then, there exist $S' \in \mathcal{C}$ and $B' \in b(\mathcal{C})$ with $|S' \cap B'| \in \{2, 3\}$.*

Proof. If $\mathcal{C}$ is binary then the result holds by Proposition 4. Otherwise, it is proved in [21] that $\mathcal{C}$ contains as a minor, say $\mathcal{D} = \mathcal{C}/I \setminus J$, one of: $\mathcal{P}_4 = \{\{1,2\},\{2,3\},\{3,4\}\}$ or $\Delta_s = \{\{1, \dots, s\} \cup \{0, i\} : i \in [s]\}$ where $s \geq 2$. Note that $\{2,3\} \in \mathcal{P}_4 \cap b(\mathcal{P}_4)$ and $\{0,1\} \in \Delta_s \cap b(\Delta_s)$. Hence, there exist $S' \in \mathcal{D}$ and $B' \in b(\mathcal{D})$ with $|S' \cap B'| = 2$. Then, for some $I' \subseteq I$ and $J' \subseteq J$ we have $S = S' \cup I \in \mathcal{C}$ and $B = B' \cup J \in b(\mathcal{C})$. In particular, $|S \cap B| = |S' \cap B'| = 2$. ☐

3.1 A restatement in terms of matroids.

To prove Proposition 4 we shall first restate the result in the language of binary matroids. Given a matroid M with a fixed element Ω we define,

$$\mathrm{port}_{\Omega}(M) = \{C \setminus \Omega : C \text{ is a circuit of } M \text{ containing } \Omega\}.$$

Binary clutters are exactly the clutters that can be expressed as ports of binary matroids. Namely,

Proposition 6 ([21]). *Let $\mathcal{C}$ be a clutter. Then the following are equivalent,*

- *$\mathcal{C}$ is a binary clutter,*
- *for some binary matroid M and $\Omega \in E(M)$, we have $\mathcal{C} = \mathrm{port}_{\Omega}(M)$.*

In addition we have the following attractive relation between matroid duality and the blocking relation.

Proposition 7 ([15] (2.A.1)). *Let M be a binary matroid and let $\Omega \in E(M)$. Then,*

$$b\big(\mathrm{port}_{\Omega}(M)\big) = \mathrm{port}_{\Omega}(M^*).$$

Here M^* denotes the dual of M. Oxley studied when a matroid contains a *quad*, namely a set of four elements that are exactly the intersection of a circuit and a cocircuit [16]. We prove the following related rooted result for binary matroids.

Proposition 8. *Let M be a binary matroid and let $\Omega \in E(M)$. Consider C and D where*

a. D is a cocircuit of M containing Ω,
b. C is a circuit of M containing Ω, and
c. $|C \cap D| \geq 4$.

Then there exists a circuit C' of M with $\Omega \in C'$ and $|C' \cap D| = 4$.

We postpone the proof until Section 3.2. Let us show that this implies our results on binary clutters.

Proof (Proposition 4). By Proposition 6, $\mathcal{C} = \mathrm{port}_{\Omega}(M^*)$ for some binary matroid M. Then $S = D \setminus \{\Omega\}$ where D is a cocircuit of M with $\Omega \in D$. By Proposition 7, $B = C \setminus \{\Omega\}$ where C is a circuit of M such that $\Omega \in C$. By Proposition 8 there exists a circuit C' of M with $\Omega \in C'$ and $|C' \cap D| = 4$. Then by Proposition 7, $B' := C' \setminus \{\Omega\}$ is a member of $b(\mathcal{C})$ where $|B' \cap S| = |(C' \cap D) \setminus \{\Omega\}| = 3$, as required. □

3.2 The proof of Proposition 8.

We shall require a few definitions first. A *signed graph* is a pair (G, Σ) where $\Sigma \subseteq E(G)$. We say that $C \subseteq E(G)$ is a *cycle* of G if C induces an Eulerian subgraph of G. It is a *polygon* if it induces a connected subgraph where every vertex has degree two. A cycle C of G is an *even cycle* of (G, Σ) if $|C \cap \Sigma|$ is even. A set of the form $\Gamma = \Sigma \triangle \delta_G(U)$ where $\delta_G(U)$ is a cut, is a *signature* of (G, Σ). Observe that (G, Σ) and (G, Γ) have the same set of even cycles.

We denote by $\mathrm{cycle}(G)$ the set of all cycles of G and by $\mathrm{ecycle}(G, \Sigma)$ the set of all even cycles of (G, Σ). The set $\mathrm{cycle}(G)$ are exactly the cycles of the graphic matroid with representation G and we identify $\mathrm{cycle}(G)$ with that matroid. The set $\mathrm{ecycle}(G, \Sigma)$ are exactly the cycles of a binary matroid that is a single lift of a graphic matroid, this is known as an even-cycle matroid [10]. We identify $\mathrm{ecycle}(G, \Sigma)$ with that matroid. We use the following folklore result that says that we can represent as signed graphs, binary matroids that are graphic after contraction of a single element. This follows readily, by considering the $0, 1$ representation of the binary matroid.

Remark 2. Let M be a binary matroid and let $e \in E(M)$ where e is not a loop. If $M/e = \mathrm{cycle}(G)$ then $M = \mathrm{ecycle}(G', \Sigma)$ where G' is obtained from G by adding loop e and $\Sigma \subseteq E(G)$ with $e \in \Sigma$.

Finally, we will need a characterization of the cocycles of even-cycle matroids.

Proposition 9 ([17] Remark 2.1). *Let (G, Σ) be a signed graph. The cocycle of* $\mathrm{ecycle}(G, \Sigma)$ *are the cuts of* G *and the signatures of* (G, Σ).

We are now ready for the principal proof in this section.

Proof (Proposition 8). We may assume $|C \cap D| \geq 6$. It suffices to prove the following weaker statement,

$$\text{There exists a circuit } C' \text{ with } \Omega \in C' \text{ and where } 4 \leq |C' \cap D| < |C \cap D|. \quad (\star)$$

Suppose for a contradiction that $(\star)$ does not hold for M with C and D satisfying (a)-(c). We assume that M among all such counterexamples minimizes $|E(M)|$.

Claim 1. Let $\hat{M} = M/I \setminus J$ where $\Omega \notin I \cup J \neq \emptyset$ and $I \cap J = \emptyset$. Then $\hat{C} = C \setminus I$ is not a circuit of $\hat{M}$ or $\hat{D} = D \setminus J$ is not a cocircuit of $\hat{M}$.

Proof of Claim. Suppose for a contradiction that $\hat{C}$ is a circuit of $\hat{M}$ and that $\hat{D}$ is a cocircuit of $\hat{M}$. As $\hat{C} \cap \hat{D} = C \cap D$, we have $\Omega \in \hat{C} \cap \hat{D}$, and $|\hat{C} \cap \hat{D}| \geq 6$. It follows that (a)-(c) hold for $\hat{M}$ with $\hat{C}, \hat{D}$. By our minimality assumption, there exists a circuit F of $\hat{M}$ with $\Omega \in F$ and $4 \leq |F \cap \hat{D}| < |\hat{C} \cap \hat{D}|$. By definition of minors, there exists $I' \subseteq I$ and $J' \subseteq J$ such that $C' = F \cup I'$ is a circuit of M and $D = \hat{D} \cup J'$. Then $C' \cap D = (F \cup I') \cap (\hat{D} \cup J') = F \cap \hat{D}$. It follows that $4 \leq |F \cap \hat{D}| = |C' \cap D|$ and $|C \cap D| = |\hat{C} \cap \hat{D}| > |F \cap \hat{D}| = |C' \cap D|$. However, then $(\star)$ holds for M, a contradiction. $\diamond$

Claim 2. $C = D$ and all elements of $E(M) \setminus C$ are in the span of C.

Proof of Claim. If there exists $e \in C \setminus D$ then let $\hat{M} = M/e$. Then $C \setminus \{e\}$ is a circuit of $\hat{M}$ and D is a cocircuit of $\hat{M}$, contradicting Claim 1. If there exists $e \in D \setminus C$ then let $M' = M \setminus e$. Then $D \setminus \{e\}$ is a cocircuit of $\hat{M}$ and C is a circuit of $\hat{M}$, contradicting Claim 1. Thus we have $C = D$. If there exists $e \notin C = D$ where e is not in the span of C, the let $\hat{M} = M/e$. Then C is a circuit of $\hat{M}$ and D is a cocircuit of $\hat{M}$, contradicting Claim 1. $\diamond$

Denote by $\Omega, e_1, \ldots, e_n$ the elements of $C = D$. Note, $n \geq 5$ since $|C \cap D| \geq 6$. Let $F = \{f_1, \ldots, f_m\}$ denote the elements in $E(M) \setminus C$. Pick $i \in [m]$. By Claim 2, there are exactly two circuits L, L' in $C \cup \{f_i\}$ and in addition $L \triangle L' = C$. Thus exactly one of L, L' contains Ω. Denote that circuit by C_i. If $|C_i| \geq 5$ then $4 \leq |C_i \cap D| < |C \cap D|$ and $(\star)$ holds with $C' = C_i$. Since circuits and cocircuits have even intersection in binary matroids, for every $i \in [m]$, C_i is a triangle (a circuit of cardinality three) with elements say, e_i, f_i, Ω.

Let $N = M/\Omega$. Then $C \setminus \{\Omega\}$ is a circuit of N. Moreover, for every $i \in [m]$, e_i, f_i are parallel in N. This implies that every element of f_i is paired with some element e_i, in particular, $n \geq m$. Let N' denote the matroid obtained from N

by replacing every parallel class e_i, f_i by e_i. Then N' consists of a single circuit $C \setminus \{\Omega\}$, in particular, it is graphic. It follows that $N = \text{cycle}(G)$ where G is the graph obtained from the polygon $C \setminus \{\Omega\}$ by adding for each $i \in [m]$ an edge f_i parallel to e_i. (G is unique up to reordering $e_1, \ldots, e_n$.)

By Remark 2, $M = \text{ecycle}(H, \Sigma)$ where H is obtained from G by adding a loop Ω and for a suitable $\Sigma \subseteq E(H)$ with $\Omega \in \Sigma$. Proposition 9 implies that the cocircuits M are either cuts of H or signatures of (H, Σ). Since D is a cocircuit of M and Ω is a loop of H, D must be a signature of (H, Σ). Since the signatures of (H, Σ) and (H, D) are the same, we will assume that $\Sigma = D$. Suppose that $n \geq m+2$. Then there exists $e_{m+1}, e_{m+2} \in D$ which are not parallel to edges in F in the graph H. It follows that $D' = \{e_{m+1}, e_{m+2}\}$ is a cut of H, hence, by Proposition 9, D' is a cocycle of M. However, $D' \subset D$, contradicting the fact that D is a cocircuit. Thus $n \leq m+1$ and $m \geq n-1 \geq 4$. In particular, there exists $f_1, f_2 \in F$ and for $i = 1, 2$, e_i, f_i are parallel in H. It follows that $C' = C \setminus \{e_1, e_2\} \cup \{f_1, f_2\}$ is a polygon of H. Moreover, $|C' \cap \Sigma|$ is even. It follows that C' is a circuit of M which satisfies ($\star$) as required. □

3.3 A geometric application.

Given a clutter $\mathcal{C}$ and $S \in \mathcal{C}$, we denote by $\chi(S)$ the characteristic vector for S, i.e. for every $e \in E(\mathcal{C})$, $\chi(S)_e = 1$ if $e \in S$ and $\chi(S)_e = 0$ otherwise. Recall that $0,1$ vertices of $P(\mathcal{C})$ are of the form $\chi(B)$ for some $B \in b(\mathcal{C})$ ([7] Remark 1.16). In this section we describe a variant of Proposition 4 that describes where the point $\chi(B')$ can be selected on the polyhedron $P(\mathcal{C})$. Namely, the following holds.

Proposition 10. *Let $\mathcal{C}$ be a binary clutter and let $\bar{z}$ be an integral corner of $P(\mathcal{C})$. Let $S \in \mathcal{C}$ for which $\sum_{e \in S} \bar{z}_e = 1$. Suppose there exists $B \in b(\mathcal{C})$ for which $|S \cap B| > 1$. Then there exists $B' \in b(\mathcal{C})$ for which $|S \cap B'| = 3$ where $\chi(B')$ is adjacent to $\bar{z}$ in $P(\mathcal{C})$.*

Before we proceed with the proof, recall the following result about tangent cones, see [18] Chapter 6, and also [23].

Proposition 11. *Let P be a polyhedron with a vertex $\bar{z}$. Let $L_1, \ldots, L_k$ be the faces of dimension 1 that contain $\bar{z}$ and for all $i \in [k]$ let v^i be a point in $L_i - \{\bar{z}\}$. Then $P \subseteq \bar{z} + \text{cone}(v^i - \bar{z} : i \in [k])$. In particular, if $y^1, \ldots, y^k$ are the vertices that are adjacent to $\bar{z}$ then*

$$P \subseteq \bar{z} + cone\big(y^i - \bar{z} : i \in [k]\big) + \text{rec}(P).$$

Here, $\text{rec}(P)$ denotes the recession cone of P.

Proof (Proposition 10). By Proposition 4 there exists $R \in b(\mathcal{C})$ for which $|S \cap R| = 3$. By Proposition 11, there exists $\lambda \in \mathbb{R}^k_+$ for which,

$$\chi(R) = \bar{z} + \sum_{i \in [k]} \lambda_i (y^i - \bar{z}). \tag{3}$$

Note, we do not need any rays of $\mathrm{rec}(P(\mathcal{C}))$ to express $\chi(R)$ in the previous equation as R is a clutter element and thus is inclusion-wise minimal. Every vertex of $P(\mathcal{C})$ is the characteristic vector of some element of the blocker ([7] Remark 1.16). Thus, for every $i \in [k]$ we have $B_i \in b(\mathcal{C})$ with $y^i = \chi(B_i)$. We also have $B_z \in b(\mathcal{B})$ with $\bar{z} = \chi(B_z)$. By construction $|S \cap B_z| = 1$ and $|S \cap B_i| \geq 1$ for all $i \in [k]$. Left multiplying (3) by $\chi(S)^\top$ yields,

$$\begin{aligned} |S \cap R| = \chi(S)^\top \chi(R) &= \chi(S)^\top \bar{z} + \sum_{i \in [k]} \lambda_i \chi(S)^\top (y^i - \bar{z}) \\ &= |S \cap B_z| + \sum_{i \in [k]} \lambda_i (|S \cap B_i| - |S \cap B_z|) \\ &= 1 + \sum_{i \in [k]} \lambda_i (|S \cap B_i| - 1). \end{aligned} \tag{4}$$

Since $|S \cap R| > 1$ it follows from (4) that for some $\kappa \in [k]$ we have $\lambda_\kappa > 0$ and $|S \cap B_\kappa| > 1$. Moreover, (3) implies that R equals the support of $\bar{z} + \sum_{i \in [k]} \lambda_i (y^i - \bar{z})$. Therefore, $R \supseteq B_\kappa \setminus B_z$ and in particular,

$$3 = |S \cap R| \geq |S \cap B_\kappa| - |S \cap B_z| = |S \cap B_\kappa| - 1.$$

Thus $|S \cap B_\kappa| \leq 4$. Since $\mathcal{C}$ is binary, we must have $|S \cap B_\kappa| \leq 3$. However, then $B' = B_\kappa$ is as required. □

4 The proof of Theorem 2.

4.1 A general tool.

We shall state a result from [12] that gives a sufficient condition for a linear program to have a dyadic solution. We require a number of definitions. Consider $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ and let P be the polyhedron $\{x : Ax \geq b\}$. Let F be a face of P. Denote by $I(F)$ the set of row indices of A corresponding to the tight constraints for F, i.e. $i \in I(F)$ if and only if $\mathrm{row}_i(A)x = b_i$ for every $x \in F$. A face F of P is *compliant* if for every $w \in \mathbb{Z}^n$ for which F is the w-optimal face, the following linear program has an optimal solution that is dyadic,

$$\min\{b^\top y : A^\top y = w, y \geq \mathbf{0}\}. \tag{5}$$

A sequence of faces of P, $(F_0, F_1, \ldots, F_k)$ is a *face-chain rooted at* F_0 if for every $\ell \in [k]$, F_ℓ is an immediate superface of $F_{\ell-1}$. This face-chain is *well-marked* if

F_k is compliant and for every $\ell \in [k]$ there exists a vector $\rho \in \mathbb{Z}^n$ and an index $\hat{\imath} \in I(F_\ell)$ such that, [2]

(g1) $\rho \in \mathrm{aff}(F_\ell) \setminus \mathrm{aff}(F_{\ell-1})$ and
(g2) $row_{\hat{\imath}}(A)\rho - b_{\hat{\imath}} = \pm 2^p$ for some $p \in \mathbb{Z}_+$.

Here is our key tool from [12], it is a combination of Theorem 2.2 and Proposition 3.1 in a different language but without essential changes to the proof. For the sake of completeness we will give a self-contained proof in the full version of this article.

Theorem 3. *Let $P = \{x : Ax \geq b\} \subset \mathbb{R}^n$ be a polyhedron where all entries of A, b are integer. Let $\bar{w} \in \mathbb{Z}^n$ and let $\bar{F}$ denote the $\bar{w}$-optimal face of P where $\bar{F} \cap \mathbb{Z}^n \neq \emptyset$. If there exists a well-marked face-chain rooted at $\bar{F}$, then there exists an optimal solution of (5) for $w = \bar{w}$ that is dyadic.*

4.2 The proof

We are now ready for the proof of our key result.

Proof (Theorem 2). Let $n = |E(\mathcal{C})|$ and let $p = |\mathcal{C}|$. Let M be the matrix where the rows correspond to the characteristic vectors of $\mathcal{C}$. Let A be the matrix obtained from M by adding as additional rows the identity matrix I_n. Then A is an $m \times n$ matrix where $m = p + n$. Let b be the vector with m entries where $b_i = 1$ if $i \in [p]$ and $b_i = 0$ otherwise. Then

$$P(\mathcal{C}) = \{x : Ax \geq b\}. \tag{6}$$

Let $F_0 = \bar{F}$ denote the w-optimal face of $P(\mathcal{C})$. Let us choose $k \in \mathbb{Z}_+$ maximum for which $(F_0, F_1, \ldots, F_k)$ is a face-chain rooted at $\bar{F}$ with the property that for every $\ell \in [k]$, there exists $\rho \in \mathbb{Z}^n$ and $\hat{\imath} \in I(F_\ell)$ such that properties (g1), (g2) are satisfied (such a chain always exists as we can pick $k = 0$).

If F_k is compliant then $(F_0, F_1, \ldots, F_k)$ is a well-marked face-chain and it follows from Theorem 3 that (5) has an optimal dyadic solution as required. Furthermore, we leave it as an exercise to prove that (5) has an optimal dyadic solution if and only if (2) does. Thus we will assume that F_k is not compliant, i.e., there exists $w' \in \mathbb{Z}_+^{E(\mathcal{C})}$ for which F_k is the w'-optimal face and (5) does not have an optimal dyadic solution for right-hand-side w'. Since $\bar{F} \subset F_k$, $v \in F_k$ and every w-minimal cover B is a w'-minimal cover, in particular, w' is minimal. Then by the fact[3] that (2) has a half-integer solution, when there are no more than 6 active dual variables and the clutter is binary, $\sigma(F_k) \geq 7$ and by hypothesis F_k is v-compatible.

[2] Note that ρ and $\hat{\imath}$ depend on ℓ.
[3] a proof is in the full version of the article.

Since w' is minimal, no constraint $x_j \geq 0$ is tight for F_k. Hence, for some $\mathcal{D} \subseteq \mathcal{C}$ the tight constraints for F_k are $x(D) = 1$ for all $D \in \mathcal{D}$. Let $\mathcal{L}$ denote the set of faces of dimension 1 containing v.

Claim 1. For every $D \in \mathcal{D}$ there exists $L \in \mathcal{L}$ and $\rho \in L \cap Z^n$ with

$$\sum_{j \in D} \rho_j \in \{2, 3\}.$$

Proof of Claim. Consider the case where there exists $B \in b(\mathcal{C})$ with $|B \cap D| > 1$. By Proposition 10 there exists $B' \in b(\mathcal{C})$ with $|D \cap B'| = 3$ and where $\rho = \chi(B')$ is adjacent to v. Let $L \in \mathcal{L}$ be the 1-dimensional face containing v and $\chi(B')$. Then L, ρ are as required. Thus we may assume that every $B \in b(\mathcal{C})$ satisfies $|B \cap D| = 1$. It can be shown that all line segments in $\mathcal{L}$ are tight for $x(D) \geq 1$ for every $D \in \mathcal{D}$. Since $P(\mathcal{C})$ is up-monotone, and by Proposition 11, some half-line $L \in \mathcal{L}$ where $L = \{v + \lambda \mathbf{1}_j : \lambda \geq 0\} \in \mathcal{L}$ is not tight for some $x(D) \geq 1$ where $D \in \mathcal{D}$. Then, $j \in D$ and $L, \rho = v + \mathbf{1}_j$ are as required. $\diamond$

Construct $\mathcal{L}' \subseteq \mathcal{L}$ as follows: for every $D \in \mathcal{D}$, include the face L in Claim 1. Observe that by construction $\mathcal{L}'$ is (F, v)-saturating. Since F_k is v-compatible, there exists $L \in \mathcal{L}'$ for which $L \cap F_{k+1} \setminus \{v\} \neq \emptyset$ where F_{k+1} is some immediate superface of F_k. L arises in Claim 1 and consider the corresponding constraint $x(D) \geq 1$ and the corresponding $\rho \in L \cap Z^n$. Denote by $\hat{\imath}$ the index of A corresponding to constraint $x(D) \geq 1$. Then (g1), (g2) hold for $\ell = k + 1$ and $(F_0, \ldots, F_{k+1})$ contradicts our choice of k. $\square$

5 Closing remarks.

In our proof for the binary case, we used the fact that if the number of active dual variables is bounded by 6 then the dual (2) has a $\frac{1}{2}$-integer optimal solution. Suppose that we could improve the result from 6 to some larger value α and only guarantee a dyadic optimal solution of (2). This would immediately allow us to replace 7 in Theorem 2 by $\alpha + 1$. This in turn, using Proposition 2, would allow us to replace 6 by α in Theorem 1. So, we in fact proved the following statement.

Theorem 4. *If Conjecture 2 holds for all binary instances with at most α active constraints, then it also holds for all binary instances where the corner is (k, ℓ)-degenerate for $k(\ell + 1) \leq \alpha$.*

In other words, proving the conjecture for a higher number of active constraints implies the conjecture for more degenerate corners.

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References

1. Abdi, A., Cornuéjols, G., Palion,Z.:. On dyadic fractional packings of T-joins. SIAM Journal on Discrete Mathematics **36**(3), 2445–2451 (2022)
2. Abdi, A., Cornuéjols, G., Guenin, B., Tunçel, L.: Dyadic linear programming and extensions. Math. Program. **213**, 473–516 (2025)
3. Abdi, A., Cornuéjols, G., Guenin, B., Tunçel, L.: Total dual dyadicness and dyadic generating sets. Math. Program. **206**, 125–143 (2024)
4. Abdi, A., Cornuéjols, G., Liu, S., Silina., O.: Strongly connected orientations and integer lattices. International Conference on Integer Programming and Combinatorial Optimization, 1–14. Cham: Springer Nature Switzerland. (2025)
5. Chvátal, V.: On certain polytopes associated with graphs. Journal of Combinatorial Theory, Series B **18**, 138–154 (1975)
6. Cornuéjols, G., Karahalios, A., Patil, V.: Computing a Nonnegative Dyadic Solution to a System of Linear Equations. Manuscript, Tepper School of Business, Carnegie Mellon University (2025)
7. Cornuéjols, G.: Combinatorial optimization: Packing and covering. Society for Industrial and Applied Mathematics (2001)
8. Edmonds, J., Fulkerson, D. R.: Bottleneck extrema. J. Combinatorial Theory **8**, 299–306, (1970)
9. Geelen, J. F., Guenin, B.: Packing odd-circuits in Eulerian graphs. J. Comb. Theory B **86**, 280–295 (2002)
10. Guenin, B., Heo, C.: Recognizing even-cycle and even-cut matroids. Math. Program. **202**, 515–542 (2023)
11. B. Guenin, B., Hwang, S.: Dyadic packing of dijoins. SIAM Journal on Discrete Mathematics **39**, 593–606 (2025)
12. Guenin, B., Tunçel, L.: Generalizations of Total Dual Integrality. arXiv preprint arXiv:2503.07925 (2025)
13. Lehman, A.: A solution of the Shannon switching game. J. Soc. Indust. Appl. Math. **12**, 687–725 (1964)
14. Lucchesi, C. L., Younger, D. H.. A minimax theorem for directed graphs. Journal of the London Mathematical Society **17**(3), 369–374 (1978)
15. Novick, B., Sebő, A.: On combinatorial properties of binary spaces. In: Balas, E., Clausen, J. (eds.) Integer Programming and Combinatorial Optimization. IPCO Lecture Notes in Computer Science, vol 920. Springer, Berlin, Heidelberg (1995)
16. Oxley, J. G.: On the intersections of circuits and cocircuits in matroids. Combinatorica **4**, 187–195 (1984)

17. Pivotto, I.: Even cycle and even cut matroids. Doctoral dissertation, University of Waterloo (2011)
18. Rockafellar, R. T., Wets, R. J.-B.: Variational analysis. Grundlehren Math. Wiss., 317 [Fundamental Principles of Mathematical Sciences] Springer-Verlag, Berlin (1998)
19. Schrijver, A.: *Combinatorial optimization. Polyhedra and efficiency. Vol. C*, volume 24,C of *Algorithms and Combinatorics*. Springer-Verlag, Berlin (2003) Disjoint paths, hypergraphs, Chapters 70–83.
20. Sebő, A.: Personal communication, February 2026.
21. Seymour, P. D.: The forbidden minors of binary clutters. J. London Math. Soc. **(2)**, 356–360 (1976)
22. Seymour, P. D.: A two-commodity cut theorem. Discrete Math. **23**, 177–181 (1978)
23. Ziegler, G. M.: Lectures on polytopes. Grad. Texts in Math., 152 Springer-Verlag, New York (1995)