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APPROXIMATING GRAPHIC TSP BY b-MATCHINGS

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Abstract. We give a polynomial-time algorithm for graphic TSP that returns a tour of length at most $\frac{89}{64} = \frac{7}{5} - \frac{3}{320}$ times the value of the subtour relaxation. This improves the long-standing ratio $\frac{7}{5}$ of Sebő and Vygen and shows that the integrality gap of the subtour LP on graphic instances is at most $\frac{89}{64} \approx 1.3906$. We optimize a non-linear objective function over an odd-join polytope, which uses (i) a new removable system of edges, and (ii) reconnection of subtours if the graph is disconnected. This approach gives a $\frac{13}{10}$-approximation algorithm for 4-regular 4-vertex-connected graphs, improving the $\frac{4}{3}$ of Newman, and a $\frac{4}{3}$-approximation for k-vertex-connected graphs of maximum degree at most $k+1$, for every $k \geq 3$. For the general graphic TSP, we construct a feasible point in the odd-join polytope, using an optimal vertex from the Subtour linear programming relaxation of TSP, which helps us bound the approximation factor by 89/64.

1. Introduction

For more than a decade, the best approximation ratio known for the graphic Traveling Salesperson Problem (graphic TSP) has been the 7/5 of Sebő and Vygen [9]. We improve it to $7/5 - 3/320$. In graphic TSP, the input is a connected unweighted graph $G = (V, E)$. The distance between two vertices is the number of edges in a shortest path between these two vertices in G. Then the goal is to find a shortest Hamiltonian circuit with respect to these distances. Equivalently, we want a connected spanning Eulerian multigraph using copies of edges of G, with length equal to its number of edge copies. Write $n = |V|$, $m = |E|$, and let $\textsc{SubtourLP}(G)$ denote the optimal value of the graphic subtour relaxation:

$$\textsc{SubtourLP}(G) := \min\big\{x(E) : x \in \mathbb{R}^E_+, \quad x(\delta_G(U)) \geq 2 \text{ for every } \varnothing \neq U \subsetneq V\big\}. \tag{1}$$

Here, $\delta_G(U)$ is the set of edges with exactly one endpoint in U, and $x(F) := \sum_{e \in F} x_e$.

Christofides [6] gave a 3/2-approximation for metric TSP in 1976. For general graphic TSP, Oveis Gharan, Saberi, and Singh [7] first improved this guarantee to $3/2 - \varepsilon$ for an absolute constant $\varepsilon > 0$. Mömke and Svensson [1] then obtained a 1.461-approximation by allowing edges to be deleted as well as doubled. Mucha [4] improved their circulation analysis to achieve the ratio 13/9, and Sebő and Vygen [9] obtained 7/5 using ear decompositions and matroid intersection methods. The 4/3 conjecture for the subtour relaxation, stated by Goemans [8, Conjecture 1], remains open even for graphic instances.

Our approach is straightforward, and strengthens the removable-pairing framework of Mömke and Svensson [1] in two ways: (i) we generalize removable pairs to removable stars, each consisting of edges incident to a common vertex. These stars are edge-disjoint, and we may delete edges from them simultaneously while preserving connectivity, provided at least one edge remains in each star; (ii) we optimize parity correction and deletions together over a degree-bounded T-join

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TABLE 1. Approximation factors for simple graphs.

Graph class	Previous Best	Our Results
Connected simple graphs	$7/5 = 1.4$ [9]	$89/64 = 1.390625$
4-regular, 4-vertex-connected	$4/3 \approx 1.3333$ [5]	$13/10 = 1.3$
3-vertex-connected, $\Delta \leq 4$	$25/18 \approx 1.3889$ [5]†	$4/3 \approx 1.3333$
k-vertex-connected, $k \geq 4$, $\Delta \leq k+1$	$7/5 = 1.4$ [9]	$4/3 \approx 1.3333$

† A variant of the argument in [5] also gives 4/3.

polytope (rather than relying on circulations) using a StarDeletionLP. We further propose a StarReconnectionLP that allows the join to select an entire star with a cost of two, which then allows us to restore connectivity by adding two copies of one of its edges. The latter allows us to push our approximation bounds beyond what was previously known.

Our key results are as follows:

(i) We give a $89/64 = 1.390625$ approximation for Graphic TSP, by proposing a b-matching based algorithm StarMatch. This is the first improvement on 7/5-approximation in 14 years [9]. Our analysis uses Star removable systems of edges, and improves on Mucha's $13/9 \approx 1.444$ guarantee for the circulation method of Mömke-Svensson [4].

(ii) We also obtain stronger bounds for several classes of highly connected graphs:
- For 4-regular 4-vertex-connected graphs, we give the first 1.3-approximation, improving upon the previous best bound of 4/3 by Newman [5].
- For Δ-degree-bounded k-vertex-connected graphs with $\Delta = k+1$, we give the first 4/3-approximation, improving upon the previous best-known of 1.4 [9].

We include Table 1 to summarize these results. We next give an overview of our techniques.

2. An Overview of Our Techniques

We first briefly discuss the generalization to Star removable systems, and then discuss the two linear programs we use for parity correction and connectivity, and the projection of the latter will give our final formulation.

2.1. Star Removable Systems.

The removable-pairing method of Mömke and Svensson specifies edges that can be deleted while preserving connectivity, provided at most one edge is deleted from each prescribed pair, that is defined using edges incident to the same vertex. We allow larger collections of edges incident to a common vertex. A Star removable system is a family of edge-disjoint stars, so that deleting all but one edge from each star still keeps the graph connected. We show how to construct such a system using DFS trees. See Figure 1(a) for an example, where we depict 3 stars and one can check that removal of all but one edge from each star maintains connectivity.

2.1.1. *A Parity-Correcting Formulation.*

A natural way to correct parity of each vertex, one can simply add an odd(G)-join J, where odd(G) denotes the odd-degree vertices of G, to the edge set E of G. The resulting multigraph is connected and has even degrees. One can then delete twice the maximum co-graphic rank r^* of J, by deleting doubled edges, without disconnecting the graph or

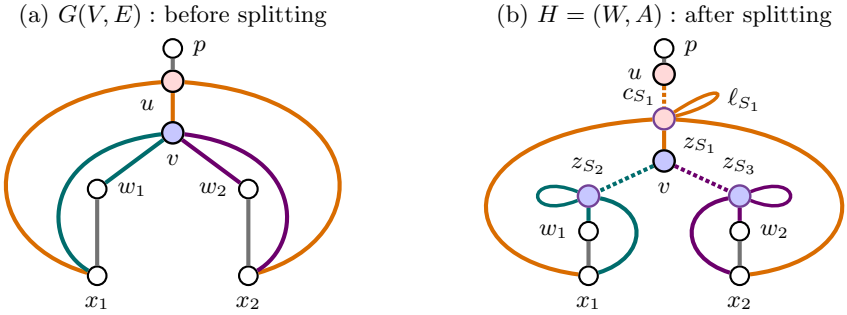


FIGURE 1. The stars $S_1 = \{uv, ux_1, ux_2\}$, $S_2 = \{vw_1, vx_1\}$, and $S_3 = \{vw_2, vx_2\}$ are orange, teal, and purple, respectively. Splitting u gives the two pink vertices u, z_{S_1}; splitting the two stars centered at v gives the three blue vertices v, z_{S_2}, z_{S_3}. Each star keeps its edge color after splitting, and its connector and loop use the same color. Straight solid edges follow the DFS tree; curved edges between distinct vertices correspond to back edges.

changing the corrected parity of the vertices. Therefore, the optimum tour of length is

$$\text{tsp}(G) = \min_{\text{odd}(G)\text{-join } J} |E| + |J| - 2r^*(J). \tag{2}$$

This *matroid-join* formulation is a constrained non-monotone supermodular function minimization, and this is NP-hard to optimize. Note that Christofides algorithm [6] first approximates $r^*(J)$ by fixing a spanning tree T', and allowing the deletion of $J \cap (E \setminus T')$; here $|J \cap (E \setminus T')| \leq r^*(J)$. Approaches that use removable systems constrain the choice of T-joins (e.g., to take at most 1 edge from each removable pair), and then approximate $|J \cap R| \leq r^*(J)$, for example, the removable system R_{MS} constructed using pairs of edges [1]. We do so by (i) giving a larger removable system $R' \supseteq R_{MS}$, and bounding the degree of the T-join J on every vertex (via the StarDeletionLP), in a way that deleting the removable edges R' in the T-join allows for local connectivity at every vertex, while ensuring global connectivity, i.e., again $|J \cap R'| \leq r^*(J)$. Since we impose these bounds only at individual vertices in an auxiliary graph (see Figure 1(b)), the degree-bounded join problem has an integral polytope formulation and any linear function can be optimized over these polytopes in polynomial time [3, pp.564-568]. We detail the construction in Appendix B. We show the following:

Theorem 2.1 (StarDeletionLP)**.** *Given a simple k-vertex connected graph G, where $k \geq 3$ and with maximum degree at most $k+1$, the* StarDeletionLP *gives a tour of length at most $4n/3 - 2/3$.*

As a comparison, we note that the algorithm of Mömke and Svensson (MS) [1, Sections 3–4] uses a circulation to choose a support graph and an MS removable pairing. On this support, they construct a distribution of joins in which each edge is selected with probability 1/3. Mucha [4] and Newman [5] use the subtour solution to improve the circulation analysis; the subsequent join distribution still has uniform marginals. Once the circulation step removes edges, the subtour solution restricted to the remaining graph need not satisfy the subtour cut constraints, making it harder to use its individual edge values in the parity correction. In our LPs, the Blossom cut inequalities give a direct connection to the subtour cut constraints, so individual subtour values can guide the join coordinates after the removable system is fixed. We construct fractional points with improved objective values, and integrality gives integral choices with at most the same objective value.

2.2. **Directly Optimizing over T-joins.** The StarDeletionLP does not allow the final tour to contain a doubled edge belonging to a removable star, and dropping at least one edge in each star may be too restrictive over T-joins. We next show how we can remove the restrictions on the T-join. We introduce a slightly modified StarReconnectionLP, which adds self-loops to our auxiliary graph and assigns weight -2 to them (Figure 1(b)). This modified auxiliary graph allows for the possibility that if the loop is not selected in the degree-bounded T-join, we will restore connectivity by adding two copies of one edge to connect the graph. In the language of the matroid-join formulation (2), we are able to (i) optimize over *all* T-joins to find a candidate join J in StarReconnectionLP, (ii) choose $J' \subseteq J \cap R$ by omitting one edge from each star fully contained in J. Then $E \setminus J'$ is connected. After deleting $J \cap R$ and doubling $J \cap N$, adding two copies of each edge in $(J \cap R) \setminus J'$ restores connectivity. This is the non-linear cost we are essentially able to bring in into the formulation. Therefore, we approximate $r^*(J)$ by $|J' \cap R| \leq r^*(J)$. The parity arguments still work, as we indeed delete all edges in $J \cap R$, while charging 2 units to each tight star (where J picks all incident edges). This extra freedom of being able to optimize over all T-joins is crucial in our fractional constructions: we can use higher values on nonremovable edges when they help satisfy the Blossom cut inequalities, while accounting for both their added cost and the cost of reconnections. We construct feasible points in StarReconnectionLPusing subtour LP vertex x^*, and prove a bound on the approximation factor. This gives us the main result of the paper.

Theorem 2.2 (General graphs)**.** *Let G be a connected simple graph on n vertices. We give a polynomial-time algorithm* StarMatch *based on the* StarReconnectionLP *that returns a tour of length*

$$\ell \leq \frac{89}{64}\textsc{SubtourLP}(G) = \left(\frac{7}{5} - \frac{3}{320}\right)\textsc{SubtourLP}(G).$$

The proof of Theorem 2.2 is given in Appendix I. Higher connectivity allows further improvements.

Theorem 2.3 (Highly connected graphs)**.** *Tours with the following worst-case guarantees can be computed in polynomial time using any DFS tree of the graph:*

(i) If G is simple, 4-regular, and 4-vertex-connected, the StarReconnectionLP *gives a tour of length at most $13n/10 - 3/5$.*

(ii) If G is simple and k-vertex-connected, where $k \geq 3$ and $\Delta(G) \leq k+1$, the StarDeletionLP *and the* StarReconnectionLP *both give a tour of length at most $4n/3 - 2/3$.*

The two parts of Theorem 2.3 are proved in Appendices G and H, respectively.

We further note that every point feasible in the StarDeletionLP extends to feasible point in the StarReconnectionLP with the same tour bound. For the Star removable system $\mathcal{S}$ constructed from T for a given graph G, with subtour optimum $\textsc{SubtourLP}(G)$, and optimum values $\textsc{StarReconnectionLP}_{\text{OPT}}$ and $\textsc{StarDeletionLP}_{\text{OPT}}$ attained by our two LPs, we show the following theorem.

Theorem 2.4 (Comparison with the circulation method; see Corollary F.5 in Appendix F.)**.** *Let G be a simple 2-vertex-connected graph on n vertices, and let T be a DFS tree rooted at a vertex r with $\deg_G(r) \geq 3$. Let $\mathcal{S}$ be the* Star *removable system constructed from T, as in Section 3. Then*

$$\begin{aligned}\text{tsp}(G) &\leq \textsc{StarReconnectionLP}_{\text{OPT}}(G, \mathcal{S}) \\ &\leq \min\left\{\textsc{StarDeletionLP}_{\text{OPT}}(G, \mathcal{S}), \min_f \text{MS}_{\text{OPT}}(f)\right\} \\ &\leq \frac{4}{3}n + \frac{2}{3}\min_f C(f) - \frac{2}{3},\end{aligned}$$

Here f ranges over minimum-cost integral Mömke–Svensson circulations for T; $\mathrm{MS}_{\mathrm{OPT}}(f)$ *is the minimum tour length obtained by optimizing the parity correction for the support and fixed MS removable pairing produced from f* [2, proof of Theorem 12.3]*, and $C(f)$ is its circulation cost* [2, Lemma 12.8].

Thus, for the same DFS tree, both of our LPs give a tour bound at least as strong as the optimized circulation bound, including the guarantees obtained by Mucha [4] and Newman [5]. In our algorithm StarMatch, we will use the same DFS tree as [1], and obtain a better guarantee using the StarReconnectionLP. In fact, in Section C.4, we give a graph G and a tree T obtained from an optimal extreme subtour solution by greedy DFS (as in StarMatch and the algorithms of Mömke–Svensson and Mucha [1, 4]). For the Star removable system $\mathcal{S}$ constructed from T, all the inequalities in C.4 are strict. We obtain

$$\begin{aligned}\textsc{SubtourLP}(G) = 26 &< \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G,\mathcal{S}) = 27\\ &< \textsc{StarDeletionLP}_{\mathrm{OPT}}(G,\mathcal{S}) = 28\\ &< \mathrm{MS}_{\mathrm{OPT}}(f^*) = 29 < \frac{4}{3}n + \frac{2}{3}C(f^*) - \frac{2}{3} = \frac{92}{3},\end{aligned}$$

here f^* is taken to be an optimal integral solution of the circulation, shown in C.4. Different choices of f^* can lead to different bounds.

2.3. **Bounding the StarReconnectionLP using the Subtour Solution.** To prove our approximation guarantee for general graphic TSP, we provide a map from the subtour LP solution x^* to a point feasible in the StarReconnectionLP. Our construction allows removable edges belonging to the DFS tree to be selected with higher probability than in the uniform join distribution of Mömke and Svensson. We use the subtour cut constraints to prove that this point satisfies the Blossom inequalities. We combine this point with a second feasible point obtained directly from the subtour solution. Using the sparsity of an optimal extreme subtour support and the structure of the DFS tree, we bound the resulting tour length and obtain our approximation guarantee.

2.4. **Outline.** Section 3 constructs Star removable systems from DFS trees. Section 4 presents the StarReconnectionLP and explains how its solutions give tours. Section 5 describes StarMatch, and Section 6 develops its analysis when $\textsc{SubtourLP}(G) = n$. Section 7 gives the 13/10 bound for 4-regular, 4-vertex-connected graphs. The appendices provide detailed proofs, including the analysis for arbitrary subtour value in Appendix I, the comparison with the Mömke–Svensson method, and the 4/3 bound for k-vertex-connected graphs of maximum degree at most $k+1$.

3. Star removable systems from a DFS tree

Let $G = (V, E)$ be a connected simple graph, with $n = |V|$ and $m = |E|$. A removable star centered at u is a nonempty set S of edges incident to u such that $G - (S \setminus \{e\})$ is connected for every $e \in S$. A Star removable system $\mathcal{S}$ is a family of pairwise edge-disjoint removable stars with the following property: for every tuple $(f_S)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} S$, obtained by choosing one edge f_S from each star S,

$$G - \bigcup_{S\in\mathcal{S}} (S \setminus \{f_S\}) \quad \text{is connected.}$$

Write $R := \bigcup_{S\in\mathcal{S}} S$, $N := E \setminus R$, and $k_S := |S|$. Thus, we may delete any subset of R that leaves at least one edge of each star; every edge of N is retained. We call the edges in R removable and those in N nonremovable, with respect to $\mathcal{S}$.

We can obtain such a Star removable system from an arbitrary DFS tree as follows. Fix a DFS tree T rooted at r, and let T_v denote the subtree rooted at v. Every edge wu outside T joins a

vertex w to a proper ancestor u and is called a back edge. For a tree edge uv, with parent u and child v, consider the set of backedges from T_v to u:

$$I(u,v) := \{wu \in E \setminus E(T) : w \in V(T_v)\}.$$

If $I(u,v) \neq \varnothing$, form the star $S(u,v) := \{uv\} \cup I(u,v)$ centered at u. This includes stars centered at the root, and different children of a vertex may give different stars. Write $\mathcal{S}$ for the family of all these stars. See Figure 9 for an example.

Lemma 3.1 (DFS Star removable system; Appendix D)**.** *Given any DFS tree T of $G = (V, E)$, the stars $\mathcal{S} = \{S(u,v) : uv \in E(T),\ I(u,v) \neq \varnothing\}$ form a* Star *removable system with $R = \bigcup_{S\in\mathcal{S}} S$. Moreover, keeping every edge of $N = E \setminus R$ and any one edge of each star gives a spanning tree.*

Every back edge belongs to exactly one star, and each star contains exactly one tree edge. Consequently,

$$|N| + |\mathcal{S}| = n - 1, \qquad m - \sum_{S\in\mathcal{S}} (k_S - 1) = n - 1. \tag{3}$$

4. The StarReconnectionLP

For an odd(G)-join J, doubling the edges of J makes every degree even. We would then like to delete both copies of each selected removable edge. If J contains an entire star, we charge the star with a cost of two units for reconnecting it back. The StarReconnectionLP includes this cost of two and chooses the parity correction to minimize the resulting tour bound.

For every star S centered at u, introduce a vertex z_S and replace each edge $uw \in S$ by the edge $z_S w$. Add the connector $c_S = uz_S$ and a loop ℓ_S at z_S. Keep every edge of N with its endpoints unchanged, and let $H = (W, A)$ be the resulting graph. See Figure 1.

Define the injective map $\mathbf{rep}\colon E(G) \to E(H)$ that simply allows us to map edges in the original graph G to the auxiliary graph H, i.e., every star edge $uw \in S$ is simply mapped to $z_S w$ in H, and non-removable edges in $e \in N$ map to their counterpart. Formally,

$$\mathrm{rep}(e) := \begin{cases} z_S w, & e = uw \in S, \quad S \text{ is centered at } u, \\ e, & e \in N. \end{cases}$$

Let S_H, R_H, and N_H be the corresponding edges in the H for S, R, and N.

The feasible region of the StarReconnectionLP is the polytope $P_{\mathcal{R}}(G, \mathcal{S})$ defined by

$$\begin{aligned} & y(S_H) + y_{c_S} + 2y_{\ell_S} \le k_S + 1 \quad (S \in \mathcal{S}) && \text{(degree bounds)}, \\ & y(C \setminus F) + |F| - y(F) \ge 1 \qquad \begin{pmatrix} U \subseteq W,\ C = \delta_H(U), \\ F \subseteq C,\ |C \setminus F| \text{ odd} \end{pmatrix} && \text{(Blossom cut inequalities)}, \\ & 0 \le y_a \le 1 \qquad (a \in E(H)) && \text{(box constraints)}. \end{aligned} \tag{4}$$

When G and $\mathcal{S}$ are fixed, we abbreviate this region as $P_{\mathcal{R}}$. The box constraints and Blossom cut inequalities define the odd(H)-join polytope. The additional degree bounds keep the polytope integral[1], which permits polynomial-time optimization via capacitated b-matching [3]. Note that loops do not cross cuts and contribute twice to degrees. For an integral point, selecting ℓ_S gives

$$y(S_H) + y_{c_S} \le k_S - 1,$$

so the join leaves an edge of S unselected and no reconnection is needed at this star. A reconnection can therefore occur only when $y_{\ell_S} = 0$, and in this case, its cost is exactly 2 units [2]. Selecting a

[1]The bound $k_S + 1$ has the same parity as $\deg_H(z_S) = k_S + 3$, which is important for the integrality of the polytope.

[2]Note that for a feasible solution to $P_{\mathcal{R}}$, $y_{\ell_S} = 0$ induces a cost of at most $2(1 - y_{\ell_S})$.

removable edge saves one copy, while selecting a nonremovable edge adds one copy. Together with the reconnection costs, this gives the following upper bound on the tour length

$$\Phi_H(y) := m - y(R_H) + y(N_H) + 2\sum_{S\in\mathcal{S}}(1 - y_{\ell_S}),$$

which can be directly obtained by optimizing the StarReconnectionLP. Note that in fact we can simplify StarReconnectionLP by factoring out the loop variables y_{ℓ_S} and simply optimize the following (projected) system using $z \in \mathbb{R}^{E(H)\setminus\{\ell_S : S\in\mathcal{S}\}}$:

$$\min_{y\in P_{\mathcal{R}}} \Phi_H(y) = \min_{y\in P_{\mathcal{R}}} m - y(R_H) + y(N_H) + 2\sum_{S\in\mathcal{S}}(1 - y_{\ell_S}) \tag{5}$$

$$= \min_{z\in \text{odd}(H^\circ)\text{-join polytope}} n - 1 + z(N_H) + \sum_{S\in\mathcal{S}} \max\{z_{c_S},\, k_S - 1 - z(S_H)\} \tag{6}$$

$$=: \min_{z\in \text{odd}(H^\circ)\text{-join polytope}} \Phi_{G,\mathcal{S}}(z), \tag{7}$$

where H° is the graph H without the self-loops. We will use $\mathcal{P}^\circ$ to denote the odd$(H^\circ)-$join polytope.

Theorem 4.1. *We can compute a tour of G with length at most $\min_{z\in\mathcal{P}^\circ}\Phi_{G,\mathcal{S}}(z)$ in polynomial time. In particular,*

$$\text{tsp}(G) \le \Phi_{G,\mathcal{S}}(z) \qquad \textit{for every } z \in \mathcal{P}^\circ.$$

The proof is given in Appendix C. The only thing that we need to do is to bound $\min_{z\in\mathcal{P}^\circ}\Phi_{G,\mathcal{S}}(z)$ with respect to the cost of the subtour LP.

5. Graphic TSP using StarMatch Algorithm

We now discuss our main algorithm StarMatch: Given a graph $G = (V, E)$, we compute an optimal extreme point x^* of the graphic subtour relaxation (1). It is known that the support G' of x^* has at most $2n-3$ edges, due to laminarity of the tight cuts [10]. By the standard block reduction [2, Proposition 12.1], it suffices to consider extreme points with support 2-vertex-connected. Next, same as Newman [5], we also construct a subtour feasible point $x \in [0, 1]^E$ that has the same cost as x^* and the same support. This is our starting point, which we'll refer to as the box-constrained SubtourLP solution.

Let G be the support of x, and suppose G is not a cycle (otherwise done). Let r be any vertex of G with degree at least 3. Construct a greedy depth-first search tree T, rooted at r, i.e, visiting each new neighbor using largest x_e edge possible. Construct the Star removable system $\mathcal{S}$ and the corresponding auxiliary graph H as in Section 3. Solve StarReconnectionLP to obtain an optimal odd(H)-join J. Let $J' = \text{rep}^{-1}(J)$ be the projection of J onto G, obtained by contracting the connector edges. Let Z be the Eulerian multigraph obtained from G by deleting the edges in $J' \cap R$ and adding a second copy of each edge in $J' \cap N$. Finally, reconnect the components of Z using doubled edges to obtain a tour. See Algorithm StarMatch for a complete description.

To present the main ideas, we describe the construction under the assumption SubtourLP$(G) = n$. Appendix I proves the feasibility and cost estimates for arbitrary subtour value, and Proposition I.7 gives their specialization to SubtourLP$(G) = n$. The general analysis gives an approximation ratio of at most 89/64 for graphic TSP.

[3] One can improve the tour construction by first adding a second copy of every edge of J' to G, then deleting both copies of every edge in a maximal independent set of the cographic matroid of G restricted to J'. This preserves connectivity and even degrees and gives a tour no longer than the one obtained above.

Algorithm 1: StarMatch

Input : A simple graph G with $|V(G)| \geq 3$, an optimal extreme point x^* in SubtourLP relaxation (1), whose support is 2-vertex-connected.

Output : A tour of G.

Find $x \in \{\text{SubtourLP}(G) \mid x(E) = x^*(E), 0 \leq x_e \leq 1 \text{ for } e \in E\}$ on the same support as x^* and let $G = (V, E)$ be the $\text{supp}(x)$

If G is a cycle, **return** G

Let root r be any vertex of degree at least three in V

T = Greedy DFS tree rooted at r, constructed by visiting each new neighbor using largest x_e edge

Let $\mathcal{S}$ = Star removable system from T, let $R := \bigcup_{S \in \mathcal{S}} S$ and $N := E \setminus R$ `// see Section 3`

J = StarReconnectionLP$(G, \mathcal{S})$, `// J is an odd-(H) join, H is constructed from G,S`

$Z = E \cup (J' \cap N) \setminus (J' \cap R)$, where $J' = \text{rep}^{-1}(J)$ `// Note Z is Eulerian`

Let $\mathcal{T}$ be the resulting tour obtained by reconnecting Z using doubled edges[3]

return $\mathcal{T}$

6. The case SubtourLP$(G) = n$

Let x be the box-constrained SubtourLP as defined in Section 5. Let $G = (V, E)$ ($|V| = n, |E| = m$) be its support, and recall that $x \in [0, 1]^E$ and $x(E) = x^*(E) = \text{SubtourLP}(G)$. Although x need not be extreme, the extremality of x^* gives $m \leq 2n - 3$ [10]. Let T be the DFS tree rooted at r constructed by StarMatch, and let $\mathcal{S}$ be the Star removable system constructed from T as in Section 3. As noted earlier, to bound the length of the tour returned by StarMatch, it suffices to construct a feasible point $Y \in \mathcal{P}^\circ$ and bound $\Phi_{G,\mathcal{S}}(Y)$ in terms of the cost of the subtour solution x. We construct Y as a convex combination of two feasible points, $Q^{(1)}$ and $Q^{(2)}$. Compared with $Q^{(1)}$, the point $Q^{(2)}$ assigns larger values to the representatives of back edges, which helps reduce the second term in each star's maximum in (6). Its values on nonremovable edges and connectors may, however, be larger. We choose the convex combination to control both terms in each maximum. We obtain $Q^{(1)}$ from a parametric construction $Q(d)$, where $d \in \mathbb{R}_+^{\mathcal{S}}$. Each star $S \in \mathcal{S}$ contains a unique DFS tree edge, and d_S determines the increase in its weight. This flexibility is one of the ingredients behind our improvement over the Mömke–Svensson bound: their matching construction uses uniform $1/3$ marginals, whereas our extension allows the weight on the removable tree edge of each star S to increase from $1/3$ to $1/3 + \frac{2}{3}d_S$. We choose a suitable vector $\bar{d}$ and set $Q^{(1)} := Q(\bar{d})$. The point $Q^{(2)}$ is obtained directly from x, by lifting the point with weight $x_e/2$ on each tree edge and $1 - x_e/2$ on each back edge to $\mathcal{P}^\circ$. We then use the convex combination

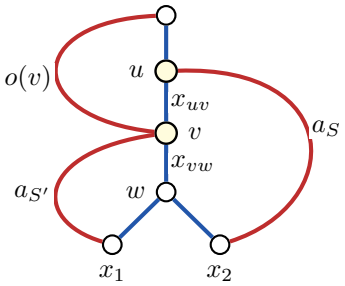
$$Y := \frac{3}{4}Q^{(1)} + \frac{1}{4}Q^{(2)} \in \mathcal{P}^\circ.$$

$Q^{(1)}$ tends to incur a higher loss on chains of certain (*positive stars*, defined in the next section) stars, that we can balance using $Q^{(2)}$. We next discuss the construction of these points.

6.1. The initial point and its excess.

Let S_r denote the unique star centered at the root r. Define the initial point $Q^{(0)} \in \mathbb{R}_+^{E(H^\circ)}$ by using box-constrained SubtourLP point x in the following way:

$$Q_a^{(0)} := \begin{cases} 1 - \dfrac{2}{3}x_b, & a = \text{rep}(b), \quad b \notin E(T) \text{ (non-tree edges)}, \\ 1, & a = c_{S_r} \text{ (connector of the root)}, \\ \dfrac{1}{3}, & \text{otherwise.} \end{cases}$$



$S = S(u, v)$, $S' = S(v, w)$

a_S: back-edge weight ending at u from T_v

$a_{S'}$: back-edge weight ending at v from T_w

$o(v)$: back-edge weight from v to its ancestors

FIGURE 2. Quantities for consecutive stars $S = S(u, v)$ and $S' = S(v, w)$ when w is the only DFS child of v. Blue edges belong to the greedy DFS tree T, and red curves represent groups of back edges. Labels give edge weights or total back edge weights under x (e.g., $o(v)$ is the sum of all back edges going between v and its proper ancestors).

It is easy to check that the point $Q^{(0)}$ belongs to the odd-join polytope $\mathcal{P}^\circ$, by checking Blossom's inequalities for an arbitrary cut (see Proposition I.7, applied with $d = 0$).

For a star $S = S(u, v)$, let $a_S := x(I(u, v))$ be the total x-value of its back edges from subtree under v to u. For each nonroot star S, its contribution to objective in (6) is

$$\max\{Q^{(0)}_{c_S}, |S| - 1 - Q^{(0)}(S_H)\} = \frac{1}{3} + \frac{2}{3}(a_S - 1)_+,$$

where $(z)_+ := \max\{z, 0\}$. Define the excess $e_S := (a_S - 1)_+$ and call S positive when $e_S > 0$. The root star contributes one, so

$$\Phi_{G,\mathcal{S}}(Q^{(0)}) = \frac{4}{3}n - \frac{2}{3} + \frac{2}{3}\sum_{S \neq S_r} e_S. \tag{8}$$

The excess therefore measures the loss above $4n/3 - 2/3$, and note that this function, for the fixed DFS tree, depends only on x. We reduce this loss by increasing the coordinates of the removable tree edges, thereby decreasing the corresponding terms $|S| - 1 - Y(S_H)$ in the objective. Note that subcubic graphs have excess 0 at every nonroot star, since at most 1 back edges enters any non-root vertex. We can therefore certify a $4n/3 - 2/3$ length tour for these graphs using $Q^{(0)}$.

6.2. **Construction of $Q(d)$.** Call a vector $d \in \mathbb{R}^{\mathcal{S}}_+$ admissible if $d_{S_r} = 0$ and $d_S \leq e_S$ for every nonroot star S. For each vertex v, let $o(v)$ be the total x value of the back edges from v to its proper ancestors in the box-constrained SUBTOURLP point. Figure 2 illustrates the relevant quantities for two consecutive stars. Our construction of $Q(d)$ will depend on the presence of consecutive stars.

For $d \in \mathbb{R}^{\mathcal{S}}_+$, define $Q(d) \in \mathbb{R}^{E(H^\circ)}_+$ as follows. For each $e \in E(G)$, set

$$Q(d)_{\mathrm{rep}(e)} := \begin{cases} \frac{1}{3} + \frac{2}{3}d_S, & e = uv, \quad S = S(u, v) \neq S_r, \\ 1 - \frac{2}{3}x_e, & e \notin E(T), \\ \frac{1}{3}, & \text{otherwise.} \end{cases} \tag{9}$$

For each star S, we further need to define the value on the connectors. We set these as:

$$Q(d)_{c_S} := \begin{cases} 1, & S = S_r, \\ \frac{1}{3} + \frac{2}{3}(d_{S^{\mathrm{par}}} - o(v))_+, & S = S(v, w), \quad v \text{ has exactly one DFS child } w, \\ & \text{and } S^{\mathrm{par}} = S(\mathrm{par}(v), v) \neq S_r \text{ exists}, \\ \frac{1}{3}, & \text{otherwise.} \end{cases} \tag{10}$$

By checking Blossom inequalities, one can verify that for every admissible $d \in \mathbb{R}_+^{\mathcal{S}}$, $Q(d)$ is in $\mathcal{P}^\circ$ (see Appendix I, Proposition I.7). Further, for an admissible vector d, the tour bound can be reduced to:

$$\Phi_{G,\mathcal{S}}(Q(d)) = \frac{4}{3}n - \frac{2}{3} + \sum_{S \neq S_r} \max\left\{ Q(d)_{c_S} - \frac{1}{3}, \frac{2}{3}(e_S - d_S) \right\}. \tag{11}$$

Note that for a nonroot star $S = S(u, v)$, increasing d_S reduces the term $e_S - d_S$, which helps the objective $\Phi_{G,\mathcal{S}}(Q(d))$. However, if there is a consecutive star S' centered at v (see Figure 2) and if v has exactly one child, then choosing $d_S > o(v)$ increases the connector coordinate $Q(d)_{c_{S'}}$. Therefore, increasing d_S on stars that belong to such chains of consecutive stars may not have a monotone impact on the objective. This is why we construct a specific d such that we can bound both the terms in the sum-of-max. On the other hand, if there is no consecutive chain of stars, then indeed increasing d_S upto e_S will reduce the objective.

6.3. **Construction of $Q^{(2)}$.** We construct a point $Q^{(2)} \in \mathcal{P}^\circ$ with

$$Q^{(2)}_{\mathrm{rep}(e)} := \begin{cases} \dfrac{x_e}{2}, & e \in E(T), \\ 1 - \dfrac{x_e}{2}, & e \notin E(T), \end{cases}$$

and show that there exist connector values satisfying, for every nonroot star $S = S(u, v)$,

$$Q^{(2)}_{c_S} \leq \begin{cases} \dfrac{x_{uv} + a_S}{2}, & u \text{ has odd degree in } T, \\ \min\left\{ \dfrac{x_{uv} + a_S}{2}, 1 - \dfrac{x_{uv} + a_S}{2} \right\}, & u \text{ has even degree in } T. \end{cases} \tag{12}$$

The proof idea is to show that every integral odd(G)-join can be extended to an integral odd(H°)-join by choosing 0/1 values for the connectors. Taking convex combinations then lifts every point in the odd(G)-join polytope to a point in $\mathcal{P}^\circ$, with each edge of G and its image receiving the same value (see Lemma I.4). The root connector has value one, and the bounds on the remaining connectors follow from local Blossom inequalities; see Appendix I.3.

6.4. **Construction of the Final Certificate Y.** Now we take a convex combination of $Q^{(1)} = Q(d)$ for a specific d and $Q^{(2)}$, to obtain a feasible point in $\mathcal{P}^\circ$, which will help us bound the approximation factor for graphic TSP. For each star $S = S(u, v)$, write $t_S := |I(u, v)|$ for its number of back edges. Define $\bar{d} \in \mathbb{R}_+^{\mathcal{S}}$ as follows[4]:

$$\bar{d}_S := \begin{cases} 0, & S = S_r, \\ \left(e_S - \dfrac{x_{uv}}{2} - \dfrac{t_S}{32} \right)_+, & S \neq S_r, \end{cases}$$

and set $Q^{(1)} := Q(\bar{d})$. We use the convex combination

$$Y := \frac{3}{4}Q^{(1)} + \frac{1}{4}Q^{(2)}. \tag{13}$$

The vector $\bar{d}$ is admissible, so Proposition I.7 gives $Q^{(1)} \in \mathcal{P}^\circ$, and hence $Y \in \mathcal{P}^\circ$ by convexity. This choice of $\bar{d}$ yields the following bound on each nonroot star.

Lemma 6.1 (Proposition I.7). *Every nonroot star $S = S(u, v)$ satisfies*

$$\max\{Y_{c_S}, t_S - Y(S_H)\} \leq \frac{1}{4} + \frac{x_{uv} + a_S}{8} + \frac{t_S}{64}.$$

[4]The Lean-verified analysis uses a different choice of $\bar{d}$ to obtain stronger bounds.

The proof of the above lemma is algebraic, and we include in Appendix I. We next use the choice of box-constrained SubtourLP point x and the size of the support of optimal SubtourLP x^* to claim that $\sum_{S\in\mathcal{S}\setminus\{S_r\}} t_S \le n-4$.

By applying Lemma 6.1, we obtain

$$\begin{aligned}\Phi_{G,\mathcal{S}}(Y) &= n-1+Y(N_H)+\sum_{S\in\mathcal{S}}\max\{Y_{c_S},\, k_S-1-Y(S_H)\}\\ &\le n+\underbrace{Y(N_H)+\sum_{S\neq S_r}\left(\frac{1}{4}+\frac{x_{uv}+a_S}{8}\right)}_{=\frac{3}{8}(n-2)}+\frac{1}{64}\underbrace{\sum_{S\neq S_r} t_S}_{\le n-4}\\ &\le n+\frac{3}{8}(n-2)+\frac{n-4}{64}=\frac{89n-52}{64}.\end{aligned}$$

The details are given in Appendix I, with the specialization to $L=n$ in Proposition I.7. Since $Y\in\mathcal{P}^\circ$, StarMatch returns a tour satisfying this bound.

For arbitrary subtour values ($\text{SubtourLP}(G)>n$), Appendix I imposes the additional restriction $d_S\le 1-x_{uv}$ for $S=S(u,v)$ and chooses the connector coordinates of $Q(d)$ using minimum cuts. The construction of $Q^{(2)}$ remains unchanged. A suitable convex combination of $Q(d)$ and $Q^{(2)}$ certifies a tour of length at most

$$\frac{89}{64}n+\frac{9}{8}\big(\text{SubtourLP}(G)-n\big)-\frac{13}{16}\le\frac{89}{64}\text{SubtourLP}(G),$$

obtained by StarMatch, which proves Theorem 2.2.

7. 4-regular, 4-connected graphs

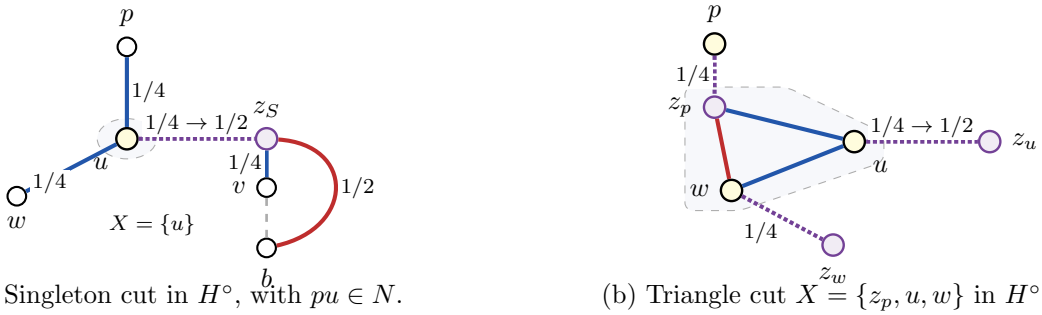
Let $G=(V,E)$ be a simple, 4-regular, 4-vertex-connected graph, and let $n:=|V|$. Fix any DFS tree T, rooted at r, and let $\mathcal{S}$ be the Star removable system constructed from T as in Section 3. We use the same notation H° for the auxiliary graph and $\Phi_{G,\mathcal{S}}$ for the objective. Our goal is to construct a point in $\mathcal{P}^\circ$ whose value under $\Phi_{G,\mathcal{S}}$ is at most $13n/10-3/5$. We use S_r for the unique star at the root. Each nonroot star has size two or three, and the root star has size four. Define the initial vector $x^{\text{init}}\in\mathbb{R}_+^{E(H^\circ)}$ by setting $x^{\text{init}}_e:=1/4$ for every nonremovable edge $e\in N_H$ and assigning the following values on the stars:

	edge rep(uv) of $S(u,v)$	edge rep(b), $b\in I(u,v)$	connector
$\lvert S\rvert=2,\ S\neq S_r$	1/4	1/2	1/4
$\lvert S\rvert=3,\ S\neq S_r$	1/4	3/4	1/4
$S=S_r$	1/4	3/4	1

These values need not satisfy every Blossom inequality. We correct them by increasing selected nonloop edge values from 1/4 to 1/2, and denote the vector after all corrections by $\bar{x}$. Our goal is to bound the number of increases that raise the objective expression in (6). The initial vector satisfies $x^{\text{init}}(S_H)+x^{\text{init}}_{c_S}\ge k_S-1$ for every star. This inequality is preserved by every increase, so the objective expression in (6) for the final vector $\bar{x}$ reduces to

$$n-1+\bar{x}(N_H)+\sum_{S\in\mathcal{S}}\bar{x}_{c_S}.$$

For x^{init}, the expression equals $5n/4-1/2$. Increasing the coordinate of rep(uv) from 1/4 to 1/2 for a removable tree edge uv leaves the objective unchanged. The same increase on an edge of N_H



(a) Singleton cut in H°, with $pu \in N$.

(b) Triangle cut $X = \{z_p, u, w\}$ in H°.

FIGURE 3. Two cuts whose Blossom inequalities are violated by x^{init}, both drawn in H°. Edge labels give the values of x^{init}; on each marked edge e, the label $1/4 \to 1/2$ shows the increase from x^{init}_e to $\bar{x}_e$. Each cut initially has three boundary edges of value $1/4$; the marked increase raises their sum to one. Shading identifies the vertices in X. Blue and red edges correspond to tree and back edges of G, respectively, via rep; dotted violet edges are connectors. The gray dashed segment in (a) represents an omitted path in H°.

or a connector raises it by $1/4$; we call this a *paid increase*, as the objective value increases with this change.

We next show that x^{init} does not satisfy the Blossom inequalities for the following two types of cuts in H°, and describe how to modify it to obtain a point in $\mathcal{P}^\circ$. Note that, after the reductions in Appendix G, cuts of size at least four satisfy all the Blossom inequalities. Full proofs are given in Appendix G.

A cut around one vertex. In G, let u be a nonroot vertex with two DFS children v, w and exactly one back edge ub ending at u, where $b \in V(T_v) \setminus \{v\}$. Write $p = \text{par}(u)$ and $S := \{uv, ub\}$ for the two-edge star centered at u. In H°, the edges of S become $z_S v$ and $z_S b$, and the connector is $c_S = uz_S$. The cut around the vertex u (in H°) is:

$$\delta_{H^\circ}(\{u\}) = \{\text{rep}(pu), \text{rep}(uw), c_S\}.$$

Each edge in this cut has value $1/4$, so $x^{\text{init}}(\delta_{H^\circ}(\{u\})) = \frac{3}{4} < 1$, violating its Blossom inequality. If pu is removable, increasing the value on $\text{rep}(pu)$ to $1/2$ satisfies this inequality without a paid increase. Otherwise, all three available increases are paid. Figure 3(a) shows the choice of increasing the connector c_S.

A cut around a triangle. Let u and w be as in the preceding case, and suppose that p, u, w form a triangle in G, where $p = \text{par}(u) \neq r$ and $u = \text{par}(w)$. Thus pu and uw are tree edges, while pw is a back edge. Suppose further that the star at p is $\{pu, pw\}$ and that w centers a two-edge star. In H°, the images of pu, uw, pw form a triangle on $X := \{z_p, u, w\}$, where z_p is the auxiliary vertex for the star at p. Its boundary consists of the connectors c_p, c_u, c_w of these three stars, each initially of value $1/4$. Recall that we increased $\text{rep}(pu)$ when repairing the cut around u. However, both endpoints of $\text{rep}(pu)$ lie in X, so that increase does not change the boundary sum of X. Thus the boundary sum therefore remains

$$x^{\text{init}}(\delta_{H^\circ}(X)) = x^{\text{init}}_{c_p} + x^{\text{init}}_{c_u} + x^{\text{init}}_{c_w} = \frac{3}{4} < 1,$$

violating the Blossom inequality for this cut. Increasing any one of these connector coordinates from $1/4$ to $1/2$ makes the boundary sum one, this modification is a *paid increase*.

Sharing the increases. Two of these deficient cuts may share a boundary edge, allowing one increase to satisfy both. For example, if two remaining singleton cuts are centered at adjacent vertices, increasing their common tree edge from $1/4$ to $1/2$ makes both boundary sums one. After

Graph class	Previously known	Presented here with full proofs	Lean-verified[5]
Connected simple graphs	$7/5 = 1.4$ [9]	$89/64$ $= 1.390625$	$14524/10481$ ≈ 1.38575
4-regular, 4-vertex-conn.	$4/3$ [5]	$13/10 = 1.3$	$9/7 \approx 1.28571$
Subquartic graphs	$25/18 \approx 1.3888$ [5]	—	$41944/31181$ ≈ 1.34518
3-vertex-connected, $\Delta \leq 4$	$25/18 \approx 1.3888$ [5] An argument similar to that in [5] would also give $4/3$	$4/3$	$4/3$
k-vertex-connected, $k \geq 4$, $\Delta \leq k+1$	$7/5 = 1.4$ [9]	$4/3$	$4/3$

TABLE 2. Approximation factors for simple graphs.

the increases on removable tree edges, form a graph with one node for each of these singleton and triangle cuts, and an edge whenever two cuts share such an increase. This graph is a forest. A maximum matching pairs cuts that can be handled together; each unmatched cut is handled by increasing one connector.

Appendix G shows that the increases on removable tree edges, together with at most $(n-2)/5$ paid increases chosen as above, give a feasible solution $\bar{x} \in \mathcal{P}^\circ$. Consequently,

$$\mathrm{tsp}(G) \leq \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}) \leq \Phi_{G,\mathcal{S}}(\bar{x}) \leq \frac{5}{4}n - \frac{1}{2} + \frac{1}{4}\frac{n-2}{5} = \frac{13}{10}n - \frac{3}{5}.$$

This bound holds for an arbitrarily chosen DFS tree T.

8. Discussion

We provided a new approximation for graphic TSP by essentially constructing a point in a T-join polytope using the subtour LP solution. We establish feasibility of the constructed point by relating the Blossom inequalities to the subtour cut constraints. For fixed fractional values on the nonloop edges, our degree-bounded-join formulation gives a decomposition into integral joins that minimizes how often each star is fully selected, and hence minimizes the expected reconnection charge.

Additional Evidence from Lean-Verification. Though the main ideas for StarMatch and for these approximations are human-generated and human-explained, for the last column of Table 2, we formalized the proofs in Lean 4 (v4.19.0) over Mathlib (commit c44e0c8e), to find better constructions of the admissible d vector, smarter choices of DFS trees, and other feasible points in the $\mathcal{P}^\circ$. The Lean verifies that our framework can improve the approximation bounds for Graphic TSP even further than those presented in the paper. These proofs show the potential of our techniques to bring down the approximation factor to $14524/10481 \approx 1.38575$ for graphic TSP, $9/7$ for 4-regular 4-vertex-connected graphs, $41944/31181 \approx 1.34518$ for subquartic graphs. We distill some of the techniques in Section J. However, we emphasize that the readers do not need to review these proofs, and our main results are self-contained with careful exposition in the main body (and appendix) of the paper. We include Table 2 to summarize these results.

[5]For these bounds we only sketch the proofs; complete proofs have been verified by Lean, see [36].

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Appendix A. Star removable systems and parity polytopes

A.1. Removable stars and Star removable systems.

Definition A.1 (Removable star)**.** *Let $G = (V, E)$ be a connected simple graph. A nonempty set $S \subseteq E$ is a removable star centered at v if every edge of S is incident to v and*

$$G - (S \setminus \{e\}) \quad \text{is connected for every } e \in S.$$

Definition A.2 (Star removable system)**.** *Let $G = (V, E)$ be a connected simple graph. A Star removable system in G is a family $\mathcal{S}$ of pairwise edge-disjoint removable stars such that, for every choice of an edge $e_S \in S$ for each $S \in \mathcal{S}$,*

$$G - \bigcup_{S \in \mathcal{S}} \big(S \setminus \{e_S\}\big) \quad \text{is connected.}$$

Thus, simultaneously deleting all but any one edge of each star preserves connectivity.

For a Star removable system $\mathcal{S}$, let R denote the set of removable edges:

$$R := \bigcup_{S \in \mathcal{S}} S.$$

The defining property also implies that $G - D$ is connected for every $D \subseteq R$ satisfying

$$|D \cap S| \le |S| - 1 \qquad (S \in \mathcal{S}).$$

A.2. The b-factor polytope and degree-bounded T-joins.

We recall the definitions of b-matchings and their polytopes from [3], using slightly different notation.

Let $G = (V, E)$ be an undirected multigraph. For $U \subseteq V$, let $\delta_G(U)$ denote the set of edges with exactly one endpoint in U, and write $\delta_G(v) := \delta_G(\{v\})$. Let $L_G(v)$ denote the set of loops at v. The degree of v is

$$\deg_G(v) := |\delta_G(v)| + 2|L_G(v)|,$$

so each loop counts twice. For $x \in \mathbb{R}^E$ and $F \subseteq E$, write

$$x(F) := \sum_{e \in F} x_e, \qquad \deg_x(v) := x(\delta_G(v)) + 2x(L_G(v)).$$

Fix vertex demands $b \in \mathbb{Z}_+^V$, and write $b(U) := \sum_{v \in U} b_v$ for $U \subseteq V$.

Definition A.3 (Simple b-matching and b-factor)**.** *A simple b-matching in G is a vector $z \in \{0,1\}^E$ satisfying*

$$\deg_z(v) \le b_v \quad \text{for every } v \in V.$$

It is perfect, also called a b-factor, if $\deg_z(v) = b_v$ for every $v \in V$. The b-factor polytope is

$$P_{\mathrm{PM}}(G, b) := \operatorname{conv}\big\{z \in \{0,1\}^E : \deg_z(v) = b_v \text{ for every } v \in V\big\}.$$

Theorem A.4 (Edmonds–Johnson [35])**.** *The b-factor polytope $P_{\mathrm{PM}}(G, b)$ consists precisely of the vectors $x \in \mathbb{R}^E$ satisfying*

$$\begin{aligned} 0 \le x_e \le 1 \quad & \text{for every } e \in E, \\ \deg_x(v) = b_v \quad & \text{for every } v \in V, \\ x\big(\delta_G(U) \setminus F\big) + |F| - x(F) \ge 1 \quad & \text{for every } U \subseteq V,\ F \subseteq \delta_G(U) \\ & \text{with } b(U) + |F| \text{ odd.} \end{aligned}$$

The last family consists of the blossom inequalities. In particular, every extreme point belongs to $\{0,1\}^E$.

Degree-bounded T-joins. For a subset F of a finite ground set, write $\mathbf{1}_F$ for its incidence vector, with coordinate $(\mathbf{1}_F)_e = 1$ exactly when $e \in F$. We use $\mathbf{1}_{\{\mathcal{E}\}}$ for the scalar indicator of a condition $\mathcal{E}$. Let $T \subseteq V$ have even cardinality. For $J \subseteq E$, write

$$\deg_J(v) := \deg_{\mathbf{1}_J}(v), \qquad \operatorname{odd}(J) := \{v \in V : \deg_J(v) \text{ is odd}\}.$$

An edge set $J \subseteq E$ is a T-join if $\operatorname{odd}(J) = T$.

For a graph $G = (V, E)$, an $\operatorname{odd}(G)$-join is an edge set $J \subseteq E$ such that $\operatorname{odd}(J) = \operatorname{odd}(G)$. The $\operatorname{odd}(G)$-join polytope is the convex hull in $\mathbb{R}^E$ of the incidence vectors of these joins.

For each $v \in V$, fix an integer degree bound $b_v \in \{0, \dots, \deg_G(v)\}$. Define

$$P_{\text{join}}(G, T, b) := \operatorname{conv}\{\mathbf{1}_J : J \subseteq E,\ \operatorname{odd}(J) = T,\ \deg_J(v) \le b_v \text{ for every } v \in V\}.$$

Put

$$t_v := (\mathbf{1}_T)_v, \qquad \beta_v := t_v + 2\left\lfloor \frac{b_v - t_v}{2} \right\rfloor \qquad (v \in V).$$

Write $\beta(U) := \sum_{v \in U} \beta_v$ for $U \subseteq V$. Note that $\beta_v = b_v$ whenever $b_v \equiv t_v \pmod 2$.

Theorem A.5 (Degree-bounded T-join polytope)**.** *The polytope $P_{\text{join}}(G, T, b)$ consists precisely of the vectors $x \in \mathbb{R}^E$ satisfying*

$$\begin{aligned} 0 \le x_e \le 1 \quad & \text{for every } e \in E, \\ \deg_x(v) \le \beta_v \quad & \text{for every } v \in V, \\ x\big(\delta_G(U) \setminus F\big) + |F| - x(F) \ge 1 \quad & \text{for every } U \subseteq V,\ F \subseteq \delta_G(U) \\ & \text{with } |T \cap U| + |F| \text{ odd.} \end{aligned} \tag{14}$$

In particular, this system is integral, and every extreme point is the incidence vector of a T-join satisfying the degree bounds.

Proof. If $\beta_v < 0$, then no T-join can satisfy the degree bound at v. Thus $P_{\text{join}}(G, T, b) = \varnothing$, and the displayed system is also infeasible because it requires $0 \le \deg_x(v) \le \beta_v < 0$.

Form G^+ by adding a set $\mathcal{L}(v)$ of distinct slack loops at each $v \in V$, where

$$|\mathcal{L}(v)| = r_v := \left\lfloor \frac{\beta_v}{2} \right\rfloor,$$

and consider $P_{\text{PM}}(G^+, \beta)$. Write x for the coordinates of the edges of G and y for the coordinates of the added loops. The degree equations are

$$\deg_x(v) + 2y(\mathcal{L}(v)) = \beta_v \qquad (v \in V). \tag{15}$$

An integral factor projects to a T-join with degrees at most b. Conversely, every such T-join extends to an integral factor by selecting exactly $(\beta_v - \deg_J(v))/2$ added loops at each vertex. Consequently, projecting $P_{\text{PM}}(G^+, \beta)$ onto the coordinates indexed by $E(G)$ gives $P_{\text{join}}(G, T, b)$.

The singleton Blossom inequalities give $\deg_x(v) \ge 1$ for $v \in T$, so the degree bounds imply $0 \le \beta_v - \deg_x(v) \le 2r_v$ at every vertex; hence the added loops can supply the required degree. The added loops do not cross any cut, and $\beta(U) \equiv |T \cap U| \pmod 2$. Together with the degree equations, this shows that a vector x satisfies (14) if and only if it can be extended to a point of $P_{\text{PM}}(G^+, \beta)$. Thus (14) describes exactly $P_{\text{join}}(G, T, b)$. ∎

Polynomial-time optimization. Linear optimization over $P_{\text{join}}(G, T, b)$ reduces in polynomial time to a weighted perfect b-matching problem via the slack-loop construction above, and therefore has a polynomial-time algorithm [35].

Upper bounds on cuts. The following proposition shows that imposing upper bounds on cuts instead of degrees makes the integral optimization problem NP-hard, even for pairwise edge-disjoint cuts of size four.

Proposition A.6. *Finding a maximum-cardinality T-join subject to*

$$|J \cap C| \le 3 \qquad (C \in \mathcal{C})$$

is NP-hard, even when $T = \varnothing$, the graph is simple and connected, and $\mathcal{C}$ consists of pairwise edge-disjoint cuts of size four.

Proof. We reduce from maximum stable set. Let G have n vertices. For each $u \in V(G)$, construct a cycle C_u of length $2n+1$. Identify one vertex of all these cycles as a common vertex r, and keep all other vertices distinct. Let H be the resulting graph.

Each cycle has n pairwise nonadjacent vertices different from r. Assign a distinct such vertex $p_{u,v}$ to each edge uv incident to u. For every $uv \in E(G)$, impose

$$|J \cap \delta_H(\{p_{u,v}, p_{v,u}\})| \le 3.$$

These cuts have size four and are pairwise edge-disjoint.

Every vertex of H other than r has degree two, so a $\varnothing$-join contains either all or none of each cycle. Conversely, every union of these cycles is a $\varnothing$-join. The constraint for uv excludes precisely the choice of both C_u and C_v. Thus feasible joins correspond to stable sets $I \subseteq V(G)$, with join cardinality $(2n+1)|I|$. The construction has polynomial size, proving the claim. ■

Appendix B. The StarDeletionLP: keeping an edge of every removable star

Let $G = (V, E)$ be a connected simple graph with $m := |E|$ edges and a Star removable system $\mathcal{S}$. For each $S \in \mathcal{S}$, let v_S denote its center and put $k_S := |S|$. Let

$$R := \bigcup_{S \in \mathcal{S}} S, \qquad N := E \setminus R.$$

We call the edges in R removable and those in N nonremovable respectively. A tour of G is a connected spanning Eulerian multigraph whose edges are copies of edges of G. Its length is its number of edge copies, and $\mathrm{tsp}(G)$ denotes the minimum length of such a tour.

We choose an $\mathrm{odd}(G)$-join J so that deleting $J \cap R$ and doubling $J \cap N$ produces a tour. To preserve connectivity, we require J to leave at least one edge of every star unselected. We express these restrictions as degree bounds in an auxiliary graph, allowing us to use the integral degree-bounded join polytope defined above.

B.1. **Constructing the auxiliary graph.** For each star S centered at v_S, introduce a vertex z_S and replace each edge $v_S u \in S$ by $z_S u$. Add the connector $c_S := v_S z_S$ and keep every edge of N with its endpoints unchanged. Let $H^{\mathrm{star}} = (W, A^{\mathrm{star}})$ be the resulting connected graph, where $W := V \cup \{z_S : S \in \mathcal{S}\}$.

Define the injective map $\mathrm{rep} : E(G) \to E(H^{\mathrm{star}})$ by

$$\mathrm{rep}(e) := \begin{cases} z_S u, & e = v_S u \in S, \quad S \text{ is centered at } v_S, \\ e, & e \in N. \end{cases}$$

Thus $\mathrm{rep}(e)$ is the edge of H^{star} obtained from e in this construction. For $B \subseteq E(G)$, write $B_{H^{\mathrm{star}}} := \{\mathrm{rep}(e) : e \in B\}$; this defines $S_{H^{\mathrm{star}}}$, $R_{H^{\mathrm{star}}}$, and $N_{H^{\mathrm{star}}}$. With $C_{H^{\mathrm{star}}} := \{c_S : S \in \mathcal{S}\}$, we have

$$A^{\mathrm{star}} = R_{H^{\mathrm{star}}} \,\dot\cup\, N_{H^{\mathrm{star}}} \,\dot\cup\, C_{H^{\mathrm{star}}}.$$

Define the vertex upper bounds

$$b_w^{\mathrm{star}} := \begin{cases} \deg_{H^{\mathrm{star}}}(w), & w \in V, \\ k_S - 1, & w = z_S, \ S \in \mathcal{S}. \end{cases} \tag{16}$$

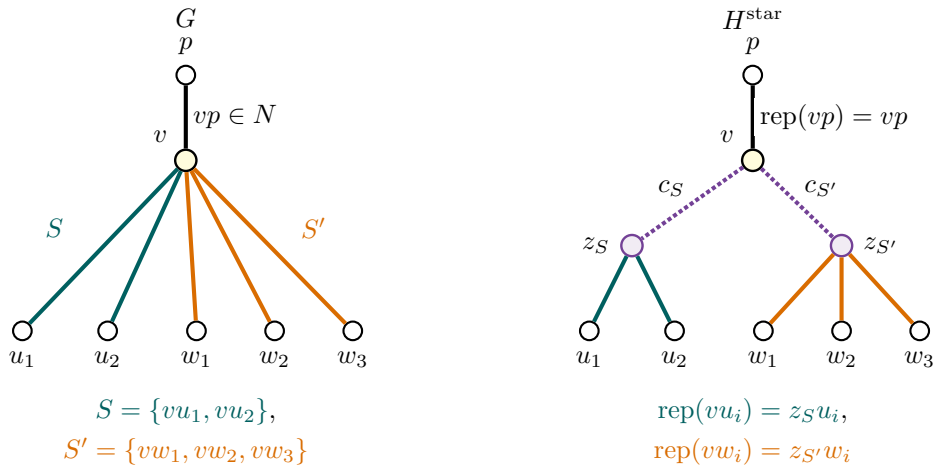


FIGURE 4. Two removable stars centered at the same vertex v. The edges of S are teal, and those of S' are orange; the same colors identify their images in H^{star}. Each star has its own auxiliary vertex and connector in H^{star}; the edge $vp \in N$ keeps its endpoints. The auxiliary vertices have degrees three and four, so their degree bounds are $|S| - 1 = 1$ and $|S'| - 1 = 2$ in StarDeletionLP, respectively.

Let

$$\mathcal{P}^{\text{SDL}}(G, \mathcal{S}) := P_{\text{join}}\big(H^{\text{star}}, \text{odd}(H^{\text{star}}), b^{\text{star}}\big).$$

By Theorem A.5, the feasible region of StarDeletionLP, abbreviated $\mathcal{P}^{\text{SDL}}$, is integral and has the following description:

$$\begin{aligned} 0 \le x_a \le 1 \qquad & \text{for every } a \in A^{\text{star}}, \\ x(S_{H^{\text{star}}}) + x_{c_S} \le k_S - 1 \qquad & \text{for every } S \in \mathcal{S}, \\ x\big(\delta_{H^{\text{star}}}(U) \setminus F\big) + |F| - x(F) \ge 1 \qquad & \text{for every } U \subseteq W,\ F \subseteq \delta_{H^{\text{star}}}(U) \\ & \text{with } |\delta_{H^{\text{star}}}(U)| + |F| \text{ odd}. \end{aligned} \tag{17}$$

Note that the degree bounds at vertices $v \in V$ follow from $0 \le x_a \le 1$, so we omit them from the displayed system.

The StarDeletionLP. Define

$$\Phi^{\text{SDL}}_{G,\mathcal{S}}(x) := m - x(R_{H^{\text{star}}}) + x(N_{H^{\text{star}}}),$$

and consider

$$\text{StarDeletionLP}_{\text{OPT}}(G, \mathcal{S}) := \min\left\{\Phi^{\text{SDL}}_{G,\mathcal{S}}(x) : x \in \mathcal{P}^{\text{SDL}}\right\}. \tag{18}$$

If $\mathcal{P}^{\text{SDL}} = \varnothing$, we set $\text{StarDeletionLP}_{\text{OPT}}(G, \mathcal{S}) := +\infty$. Thus, selecting a removable edge decreases the objective by one, selecting a nonremovable edge increases it by one, and selecting a connector leaves it unchanged.

How do we obtain a tour? By integrality, whenever the StarDeletionLP is feasible, it has an integral optimal solution z. Let

$$J := \{e \in E : z_{\text{rep}(e)} = 1\}.$$

Contracting all connectors in H^{star} gives G. Since z defines an $\text{odd}(H^{\text{star}})$-join, summing the degree parities at the vertices merged by each contraction gives

$$\text{odd}(J) = \text{odd}(G). \tag{19}$$

The star degree bounds also give

$$|J \cap S| = z(S_{H^{\text{star}}}) \le k_S - 1 \qquad (S \in \mathcal{S}).$$

Starting with G, delete the edges of $J \cap R$ and double the edges of $J \cap N$. The resulting multigraph T_z is connected because $\mathcal{S}$ is a Star removable system and at least one edge of each star remains. Every degree is even because J is an $\text{odd}(G)$-join. Thus T_z is a tour of length

$$|E(T_z)| = m - |J \cap R| + |J \cap N| = \Phi^{\text{SDL}}_{G,\mathcal{S}}(z). \tag{20}$$

Theorem B.1 (StarDeletionLP tour bound)**.** *If $\mathcal{P}^{\text{SDL}}$ is nonempty, an integral optimal solution of (18) produces a tour of length*

$$\text{StarDeletionLP}_{\text{OPT}}(G, \mathcal{S}).$$

Consequently,

$$\text{tsp}(G) \le \text{StarDeletionLP}_{\text{OPT}}(G, \mathcal{S}).$$

More generally, every $x \in \mathcal{P}^{\text{SDL}}$ certifies

$$\text{tsp}(G) \le \Phi^{\text{SDL}}_{G,\mathcal{S}}(x).$$

Proof. Theorem A.5 guarantees an integral optimal solution z. Apply (20) and use $\Phi^{\text{SDL}}_{G,\mathcal{S}}(z) = \text{StarDeletionLP}_{\text{OPT}}(G, \mathcal{S}) \le \Phi^{\text{SDL}}_{G,\mathcal{S}}(x)$. ∎

Appendix C. The StarReconnectionLP: allowing fully selected stars and restoring connectivity

As in the StarDeletionLP, we start with G, choose an $\text{odd}(G)$-join J, double its edges, and delete both copies of every edge in $J \cap R$. The StarReconnectionLP allows J to contain an entire removable star. In that case, the deletions remove every edge of the star, so we choose an arbitrary edge of that star and add back two copies of it.

After doing this for every fully selected star, at least one edge of each star remains, so the Star removable system property ensures connectivity. Adding two copies preserves the even degrees and increases the tour length by two for each such star. We include this additional cost in the StarReconnectionLP objective.

Section C.4 gives an example where the StarReconnectionLP yields a smaller tour bound for the same Star removable system. We do not know whether the StarDeletionLP gives an approximation ratio below 4/3 for simple 4-regular, 4-vertex-connected graphs. Using the StarReconnectionLP, we obtain a 13/10 bound for this class in Section G.

C.1. The auxiliary graph and degree bounds.

Construct $H = (W, A)$ from H^{star} by adding one loop ℓ_S at each z_S, so

$$A = A^{\text{star}} \dot\cup \{\ell_S : S \in \mathcal{S}\}.$$

These loops are part of the StarReconnectionLP; the slack loops in Theorem A.5 are used only in the reduction to b-factors.

Let H° be the graph obtained from H by deleting the loops ℓ_S, and write $A^\circ := E(H^\circ)$. Thus $H^\circ = H^{\text{star}}$ and $A^\circ = A^{\text{star}}$. Deleting loops preserves degree parity, so $\text{odd}(H^\circ) = \text{odd}(H)$. To distinguish the two formulations, we use H^{star} and A^{star} when working with the StarDeletionLP, and H° and A° for the nonloop graph and its edge set when working with the StarReconnectionLP. We use the same map rep, now viewed as a map from $E(G)$ to $E(H)$. Recall that rep maps each edge $uw \in S$, where S is centered at u, to $z_S w$, and each edge $e \in N$ to itself. For $B \subseteq E(G)$,

write $B_H := \{\mathrm{rep}(e) : e \in B\} = B_{H^{\mathrm{star}}}$; in particular, this defines S_H, R_H, and N_H. Write $C_H := \{c_S : S \in \mathcal{S}\}$ for the set of connectors.

Define the vertex upper bounds

$$b_w^{\mathrm{SR}} := \begin{cases} \deg_H(w), & w \in V, \\ k_S + 1, & w = z_S, \quad S \in \mathcal{S}. \end{cases} \tag{21}$$

Consider the degree-bounded T-join polytope

$$P_{\mathcal{R}}(G, \mathcal{S}) := P_{\mathrm{join}}\big(H, \mathrm{odd}(H), b^{\mathrm{SR}}\big).$$

When G and $\mathcal{S}$ are fixed, abbreviate $P_{\mathcal{R}}(G, \mathcal{S})$ to $P_{\mathcal{R}}$. Its projection onto the nonloop coordinates is denoted by

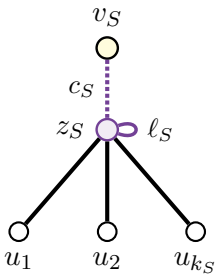
$$\mathcal{P}^\circ(G, \mathcal{S}) := \{y \in \mathbb{R}^{A^\circ} : \text{there exists } \gamma \in \mathbb{R}^{\mathcal{S}} \text{ with } (y, \gamma) \in P_{\mathcal{R}}(G, \mathcal{S})\},$$

where γ_S is the coordinate of the loop ℓ_S for $S \in \mathcal{S}$. We abbreviate this projection to $\mathcal{P}^\circ$ when G and $\mathcal{S}$ are fixed.

By Theorem A.5, $P_{\mathcal{R}}$ is integral and consists precisely of the vectors $x \in \mathbb{R}^A$ satisfying

$$\begin{aligned} 0 \le x_a \le 1 \qquad & \text{for every } a \in A, \\ x(S_H) + x_{c_S} + 2x_{\ell_S} \le k_S + 1 \qquad & \text{for every } S \in \mathcal{S}, \\ x\big(\delta_H(U) \setminus F\big) + |F| - x(F) \ge 1 \qquad & \text{for every } U \subseteq W,\ F \subseteq \delta_H(U) \\ & \text{with } |\delta_H(U)| + |F| \text{ odd}. \end{aligned} \tag{22}$$

Note that $\mathcal{P}^\circ$ is the $\mathrm{odd}(H^\circ)$-join polytope of H°. The vector in $\mathbb{R}^A$ with value one on A° and zero on every loop belongs to $P_{\mathcal{R}}$, so $P_{\mathcal{R}} \neq \varnothing$.



$x(S_H) + x_{c_S} + 2x_{\ell_S} \le k_S + 1$

ℓ_S selected $\Longrightarrow |J \cap S| \le k_S - 1$

$S \subseteq J \Longrightarrow \ell_S$ absent

FIGURE 5. The StarReconnectionLP adds one loop ℓ_S at each z_S in H. Selecting ℓ_S prevents the join from selecting every edge of S_H. If every edge of S_H is selected, we add two copies of an edge of S in G.

C.2. **The linear program.** Define

$$\Phi_H(x) := m - x(R_H) + x(N_H) + 2\sum_{S \in \mathcal{S}} (1 - x_{\ell_S}).$$

The following theorem bounds the tour length by $\Phi_H(x)$ for $x \in P_{\mathcal{R}}$. The linear program is

$$\mathrm{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}) := \min\{\Phi_H(x) : x \in P_{\mathcal{R}}\}. \tag{23}$$

Theorem C.1 (StarReconnectionLP tour bound)**.** *For every $x \in P_{\mathcal{R}}$, there is a distribution of tours of G whose expected length is at most*

$$\Phi_H(x).$$

Consequently,

$$\mathrm{tsp}(G) \le \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}).$$

Proof.

Let $x \in P_{\mathcal{R}}$. By Theorem A.5, there are edge sets $K_1, \dots, K_t \subseteq A$ and coefficients $\lambda_1, \dots, \lambda_t$ such that

$$x = \sum_{i=1}^{t} \lambda_i \mathbf{1}_{K_i}, \qquad \lambda_i > 0, \qquad \sum_{i=1}^{t} \lambda_i = 1, \tag{24}$$

where, for every i,

$$\mathrm{odd}(K_i) = \mathrm{odd}(H), \qquad \deg_{K_i}(w) \le b_w^{\mathrm{SR}} \quad (w \in W).$$

For every edge $a \in A$, the convex combination gives

$$\sum_{i=1}^{t} \lambda_i \mathbf{1}_{\{a \in K_i\}} = x_a.$$

For each i, let

$$J_i := \{e \in E : \mathrm{rep}(e) \in K_i\}.$$

Deleting loops preserves degree parity, so the projection argument used for the StarDeletionLP shows that J_i is an $\mathrm{odd}(G)$-join.

If $\ell_S \in K_i$, the degree bound at z_S gives

$$|J_i \cap S| + \mathbf{1}_{\{c_S \in K_i\}} + 2 \le k_S + 1,$$

and hence $|J_i \cap S| \le k_S - 1$. Therefore, if J_i contains every edge of S, then $\ell_S \notin K_i$. It follows that

$$\sum_{i=1}^{t} \lambda_i \mathbf{1}_{\{S \subseteq J_i\}} \le \sum_{i=1}^{t} \lambda_i \mathbf{1}_{\{\ell_S \notin K_i\}} = 1 - x_{\ell_S}. \tag{25}$$

Apply the construction above to each J_i, adding back two copies of an arbitrary edge from each star $S \subseteq J_i$. The resulting multigraph T_i is connected and has even degrees, so it is a tour of length

$$|E(T_i)| = m - |J_i \cap R| + |J_i \cap N| + 2 \sum_{S \in \mathcal{S}} \mathbf{1}_{\{S \subseteq J_i\}}.$$

Multiplying by λ_i, summing over i, and using (25), we obtain

$$\begin{aligned} \sum_{i=1}^{t} \lambda_i |E(T_i)| &= m - x(R_H) + x(N_H) + 2 \sum_{S \in \mathcal{S}} \sum_{i=1}^{t} \lambda_i \mathbf{1}_{\{S \subseteq J_i\}} \\ &\le m - x(R_H) + x(N_H) + 2 \sum_{S \in \mathcal{S}} (1 - x_{\ell_S}) \\ &= \Phi_H(x). \end{aligned}$$

At least one T_i has length no greater than this weighted average. Taking x to minimize Φ_H over $P_{\mathcal{R}}$ proves the final bound. ∎

C.3. **The objective on nonloop coordinates.** The projection $\mathcal{P}^\circ$ consists precisely of the vectors $y \in \mathbb{R}^{A^\circ}$ satisfying

$$\begin{aligned} 0 \le y_a \le 1 \quad & \text{for every } a \in A^\circ, \\ y(C \setminus F) + |F| - y(F) \ge 1 \quad & \text{for every } U \subseteq W,\ C = \delta_{H^\circ}(U),\ F \subseteq C \\ & \text{with } |C \setminus F| \text{ odd}. \end{aligned} \tag{26}$$

Every point of $P_{\mathcal{R}}$ projects to a vector satisfying these inequalities. Conversely, any such vector extends to a point of $P_{\mathcal{R}}$ by setting all loop coordinates to zero. The degree bounds then follow from $y_a \le 1$, and loops do not cross cuts.

For a fixed graph G and Star removable system $\mathcal{S}$, define

$$\Phi_{G,\mathcal{S}}(y) := \min\{\Phi_H(x) : x \in P_{\mathcal{R}}(G,\mathcal{S}),\ x|_{A^\circ} = y\} \qquad (y \in \mathcal{P}^\circ(G,\mathcal{S})).$$

We show that

$$\Phi_{G,\mathcal{S}}(y) = m - \sum_{S\in\mathcal{S}}(k_S - 1) + y(N_H) + \sum_{S\in\mathcal{S}} \max\{y_{c_S},\, k_S - 1 - y(S_H)\}. \tag{27}$$

For $y \in \mathcal{P}^\circ$, define $\widetilde{y} \in \mathbb{R}^A_+$ by

$$\widetilde{y}_a := y_a \quad (a \in A^\circ), \qquad \widetilde{y}_{\ell_S} := \min\left\{1, \frac{k_S + 1 - y(S_H) - y_{c_S}}{2}\right\} \quad (S \in \mathcal{S}).$$

These coordinates satisfy the box and degree bounds. Since loops cross no cut, the Blossom inequalities are unchanged, so $\widetilde{y} \in P_{\mathcal{R}}$. For fixed y, this choice makes each loop coordinate as large as possible and therefore minimizes Φ_H. Substituting these values and using $y(R_H) = \sum_{S\in\mathcal{S}} y(S_H)$ gives

$$\begin{aligned} \Phi_{G,\mathcal{S}}(y) &= \Phi_H(\widetilde{y}) \\ &= m + y(N_H) - \sum_{S\in\mathcal{S}} y(S_H) + \sum_{S\in\mathcal{S}} \max\{0,\, y(S_H) + y_{c_S} - k_S + 1\} \\ &= m - \sum_{S\in\mathcal{S}}(k_S - 1) + y(N_H) + \sum_{S\in\mathcal{S}} \max\{y_{c_S},\, k_S - 1 - y(S_H)\}. \end{aligned}$$

Corollary C.2. *For every $y \in \mathcal{P}^\circ$, there is a distribution of tours of G whose expected length is at most $\Phi_{G,\mathcal{S}}(y)$. Moreover,*

$$\operatorname{tsp}(G) \le \min_{y\in\mathcal{P}^\circ(G,\mathcal{S})} \Phi_{G,\mathcal{S}}(y) = \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G,\mathcal{S}).$$

Proof. Apply Theorem C.1 to $\widetilde{y}$. Every point of $P_{\mathcal{R}}$ projects to a point of $\mathcal{P}^\circ$, and the displayed choice minimizes the tour bound for each fixed projection. This proves the equality of the two optimal values. ∎

Proposition C.3 (Comparison of the two formulations)**.** *Every $x \in \mathcal{P}^{\mathrm{SDL}}$ extends to a point $\widetilde{x} \in P_{\mathcal{R}}$ by setting $\widetilde{x}_a := x_a$ for $a \in A^{\mathrm{star}}$ and $\widetilde{x}_{\ell_S} := 1$ for all $S \in \mathcal{S}$. Its certificate satisfies*

$$\Phi_H(\widetilde{x}) = \Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(x).$$

Consequently, whenever $\mathcal{P}^{\mathrm{SDL}}$ is nonempty,

$$\textsc{StarReconnectionLP}_{\mathrm{OPT}}(G,\mathcal{S}) \le \textsc{StarDeletionLP}_{\mathrm{OPT}}(G,\mathcal{S}).$$

Proof. The auxiliary degree bound increases from $k_S - 1$ to $k_S + 1$, exactly the contribution of the added loop. The parity-cut inequalities do not change, since loops belong to no cut. The added loops each have value one, so their terms $2(1 - \widetilde{x}_{\ell_S})$ are zero. Hence $\Phi_H(\widetilde{x}) = \Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(x)$. ∎

The next example shows that the StarReconnectionLP can give a strictly better tour bound than the StarDeletionLP for the same Star removable system.

C.4. **Comparing the two LPs with the circulation method.** In this comparison, both the Mömke–Svensson circulation method and StarMatch construct their DFS tree by running greedy DFS on the same optimal subtour solution. We use the same root and tie-breaking choices for both methods. The graph in Figure 6 has 23 vertices and 30 edges. The displayed vector x is a positive optimal extreme point of the subtour LP, with $x_e \in \{1/2, 1\}$ and $x(E) = 26$. We root the DFS at 3 and explore the children from left to right as drawn. The resulting stars are

$$S_0 = \{(0,2),(0,14),(0,16),(0,20)\}, \quad S_1 = \{(1,12),(1,13)\},$$
$$S_3 = \{(3,11),(2,3),(3,4)\}, \quad S_6 = \{(6,7),(5,6)\},$$
$$S_8 = \{(8,21),(8,9)\},$$

where (u, v) denotes the edge joining u and v. By Lemma D.1, $\mathcal{S} = \{S_0, S_1, S_3, S_6, S_8\}$ is a Star removable system.

For the Mömke–Svensson method, let f^* be the integral circulation displayed in Figure 8a, which minimizes $C(f)$ from (34). Write $\mathrm{MS}_{\mathrm{OPT}}(f^*)$ for the minimum tour length obtained by optimizing the parity correction on its support with the fixed MS removable pairing specified below [2, proof of Theorem 12.3]; see also (36). For these choices,

$$\begin{aligned}\textsc{SubtourLP}(G) = 26 &< \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}) = 27\\ &< \textsc{StarDeletionLP}_{\mathrm{OPT}}(G, \mathcal{S}) = 28\\ &< \mathrm{MS}_{\mathrm{OPT}}(f^*) = 29\\ &< \frac{4}{3}n + \frac{2}{3}C(f^*) - \frac{2}{3} = \frac{92}{3},\end{aligned}$$

Figure 7 shows tours attaining the two LP bounds.

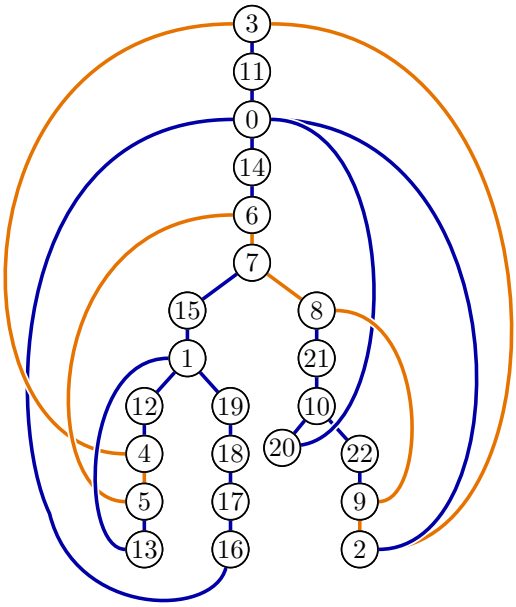
The minimum circulation cost is 1. Figure 8a shows the edges retained by f^*. For this support, use the MS removable pairs

$$\{(0,14),(0,16)\}, \qquad \{(1,12),(1,13)\}, \qquad \{(6,7),(5,6)\}.$$

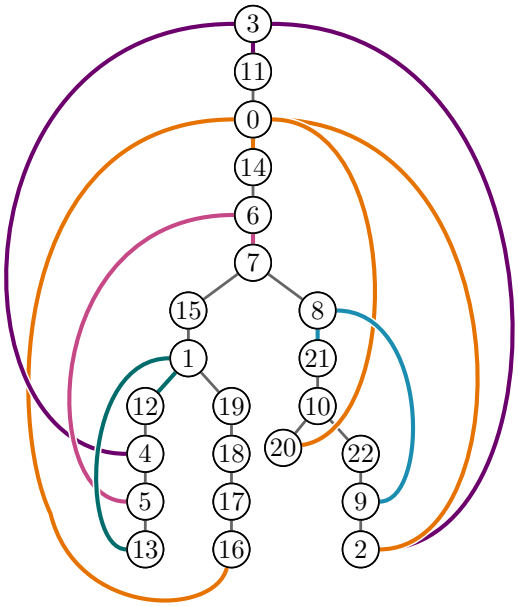
The best parity correction gives the tour of length 29 shown in Figure 8b. Table 3 also gives the standard MS bound for this circulation.

Quantity	Value
Subtour optimum $\textsc{SubtourLP}(G)$	26
StarReconnectionLP, StarMatch	27
StarDeletionLP, (18)	28
MS with optimized parity correction, $\mathrm{MS}_{\mathrm{OPT}}(f^*)$	29
MS bound, (37)	$\frac{4}{3}n + \frac{2}{3}C(f^*) - \frac{2}{3} = \frac{92}{3} \approx 30.667$

Table 3. The two LP optima for the specified DFS tree and the optimized MS tour length on the displayed support. The standard MS estimate gives an integer bound of 30.

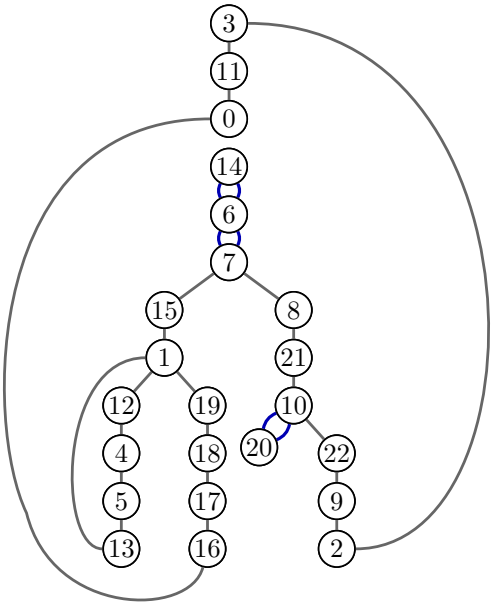


(A) Optimal extreme point, SubtourLP$(G) = 26$. Blue edges: $x_e = 1$; orange edges: $x_e = 1/2$.

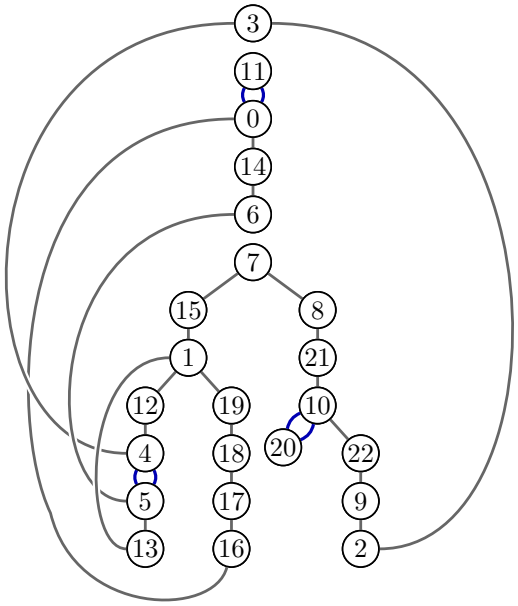


(B) The stars S_0, S_1, S_3, S_6, S_8.

FIGURE 6. The subtour solution and the removable stars for the greedy DFS rooted at 3. Straight edges belong to the DFS tree; curved edges are back edges. Each star has its own color in (b).

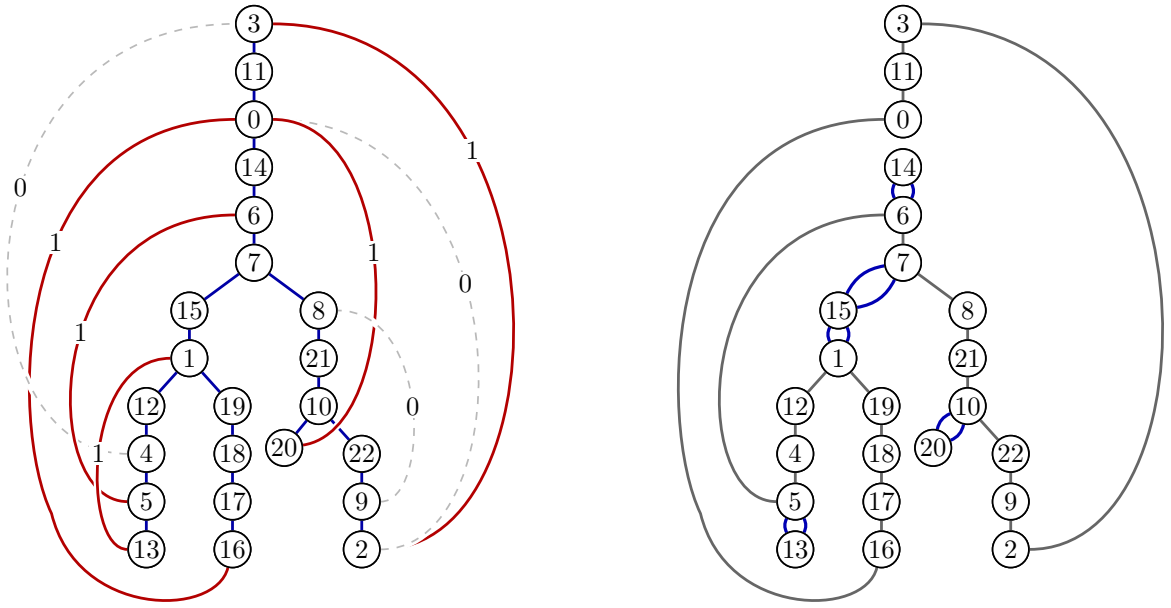


(A) StarReconnectionLP: length 27. Two copies of $(6, 7)$ reconnect S_6.



(B) StarDeletionLP: length 28. At least one edge of every star is kept.

FIGURE 7. Tours obtained from the Star removable system in Figure 6b. Blue parallel arcs denote two copies of an edge; every other displayed edge is used once.



(A) A minimum-cost circulation support. Back-arc flows are shown; $C(f^*) = 1$.

(B) MS: length 29 on the displayed support.

FIGURE 8. A minimum-cost circulation support and its optimized tour. Dashed edges are omitted from the support. The labels in (a) give back-arc flow values. In (b), blue parallel arcs denote doubled edges.

APPENDIX D. REMOVABLE STARS FROM A FIXED DFS TREE

The two LPs take a valid STAR removable system as input. This section constructs one from any fixed DFS tree.

This construction generalizes the DFS removable-pairing method of Mömke and Svensson [1, Section 4]: rather than pairing a child edge with just one back edge from that child subtree, we group it with all such back edges ending at the same parent.

Let $G = (V, E)$ be a connected simple graph, and fix a DFS tree T rooted at r. Write $n := |V|$ and let T_v denote the subtree rooted at v. Every edge outside $E(T)$ joins a vertex to one of its proper ancestors. We call these edges back edges, with upper endpoint denoting the endpoint closer to r.

For each tree edge uv, where u is the parent of v, define

$$I(u, v) := \{uw \in E \setminus E(T) : w \in V(T_v)\}. \tag{28}$$

Whenever $I(u, v) \neq \varnothing$, form

$$S(u, v) := \{uv\} \cup I(u, v), \tag{29}$$

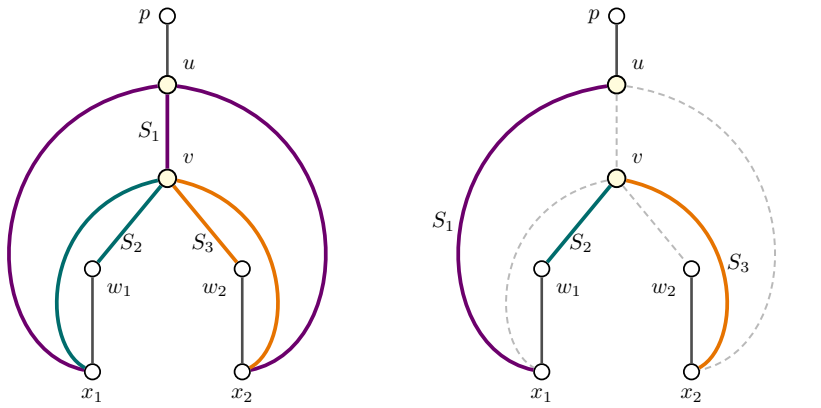
centered at u. Let $\mathcal{S}$ be the family of all these stars, and put

$$R := \bigcup_{S \in \mathcal{S}} S, \qquad N := E \setminus R.$$

In particular, all back edges belong to R, and

$$N = \{uv \in E(T) : I(u, v) = \varnothing\}.$$

The construction includes the root and allows different stars with the same center, one for each applicable child subtree.



(A) Three stars; S_2 and S_3 are centered at v.

(B) Keep ux_1, vw_1, and vx_2, one from each star.

FIGURE 9. The DFS stars $S_1 = \{uv, ux_1, ux_2\}$, $S_2 = \{vw_1, vx_1\}$, and $S_3 = \{vw_2, vx_2\}$ have different colors. Straight edges belong to the DFS tree; curved edges are back edges. On the right, gray dashed edges are deleted. The remaining edges form the spanning path $p - u - x_1 - w_1 - v - x_2 - w_2$.

Lemma D.1 (DFS removability)**.** *The family $\mathcal{S}$ is a* STAR *removable system in G. Moreover, keeping every edge of N and exactly one edge of each $S \in \mathcal{S}$ always gives a spanning tree.*

Proof. Every back edge belongs to exactly one group $I(u, v)$: its upper endpoint determines u, and its lower endpoint determines the child v on the tree path below u. Distinct groups also have distinct tree edges. Hence the stars are pairwise edge-disjoint.

Let $D \subseteq R$ satisfy $|D \cap S| \leq |S| - 1$ for every $S \in \mathcal{S}$. Enlarge D to $D' \subseteq R$ so that

$$|D' \cap S| = |S| - 1 \qquad (S \in \mathcal{S}).$$

It suffices to prove that $G - D'$ is connected.

We first prove the following claim:

(30) Every cycle of G contains two edges of the same star.

Let C be a cycle. Since T is a tree, C contains a back edge. Choose a back edge $uw \in E(C)$ whose upper endpoint u has minimum depth among the upper endpoints of all back edges in C. Let v be the child of u on the tree path from u to w. Then $uw \in I(u, v)$ crosses the cut $\delta_G(V(T_v))$. A cycle crosses every cut an even number of times, so there is another edge

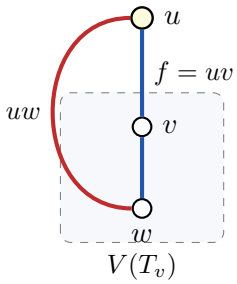
$$f \in E(C) \cap \delta_G(V(T_v)), \qquad f \neq uw.$$

We show that f also belongs to $S(u, v)$.

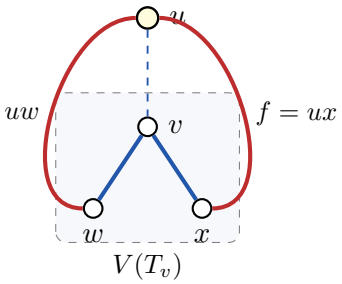
Case 1: f is a tree edge. The only tree edge crossing $\delta_G(V(T_v))$ is uv. Thus $f = uv$, and both f and uw belong to $S(u, v)$.

Case 2: f is a back edge. By the DFS ancestor property, its endpoint outside T_v is either u or a proper ancestor of u. The latter would contradict the choice of uw, since f also belongs to C. Therefore, f joins u to a vertex of T_v, so $f \in I(u, v)$. Again, f and uw belong to $S(u, v)$.

This proves (30). Figure 10 shows the two cases for f.



(A) The second crossing edge is the tree edge uv.



(B) The second crossing edge is a back edge ending at u.

FIGURE 10. The two cases in the proof of (30). The solid edges form the cycle C, and the shaded region contains $V(T_v)$. In both panels, uw and f cross its boundary and belong to $S(u, v)$. The dashed tree edge in the right panel does not belong to C.

Exactly one edge of each star remains in $G - D'$. By (30), this graph contains no cycle. Each star contains one tree edge and $|S| - 1$ back edges, and every back edge belongs to exactly one star. Hence

$$|E| = n - 1 + \sum_{S \in \mathcal{S}} (|S| - 1).$$

Since D' contains exactly $|S| - 1$ edges from each star,

$$|E(G - D')| = |E| - \sum_{S \in \mathcal{S}} (|S| - 1) = n - 1.$$

Thus $G - D'$ is a forest on n vertices with $n - 1$ edges, so it is a spanning tree. Since $D \subseteq D'$, the graph $G - D$ is also connected. Therefore, $\mathcal{S}$ is a STAR removable system. ■

APPENDIX E. THE MÖMKE–SVENSSON FRAMEWORK

In this section, we recall the Mömke–Svensson framework.

E.1. **MS removable pairings and the T-join polytope.** We follow the auxiliary-graph construction in [2, Section 12.2]. Let $G = (V, E)$ be a 2-vertex-connected simple graph. An MS removable pairing $(R_{\mathrm{MS}}, \mathcal{P})$ consists of $R_{\mathrm{MS}} \subseteq E$ and pairwise edge-disjoint pairs $P \subseteq R_{\mathrm{MS}}$, each centered at a vertex of degree at least three, such that $G - D$ is connected whenever

$$D \subseteq R_{\mathrm{MS}}, \qquad |D \cap P| \leq 1 \quad (P \in \mathcal{P}).$$

Edges of R_{MS} outside the pairs may be deleted without any pair constraint.

For each pair $P = \{vw, vw'\}$ centered at v, introduce a vertex v_P, replace vw, vw' by $v_P w, v_P w'$, and add the connector vv_P. This is the $k_S = 2$ case of Figure 4. Denote the resulting graph by Q, and the representative of $e \in E$ by e^Q. The proof of [2, Theorem 12.3] shows that Q is 2-edge-connected; each new vertex v_P has degree three.

Lemma E.1 (The one-third convex combination; [2, proof of Theorem 12.3])**.** *The vector $y := \frac{1}{3}\mathbf{1}_{E(Q)}$ belongs to the* odd(Q)*-join polytope. It admits a decomposition*

$$y = \sum_{i=1}^{t} \lambda_i \mathbf{1}_{K_i}, \qquad \lambda_i > 0, \qquad \sum_{i=1}^{t} \lambda_i = 1, \tag{31}$$

where every K_i is an odd(Q)*-join and* $|K_i \cap \delta_Q(v_P)| = 1$ *for every* $P \in \mathcal{P}$*. The projected sets* $J_i := \{e \in E : e^Q \in K_i\}$ *satisfy*

$$\text{odd}(J_i) = \text{odd}(G), \qquad |J_i \cap P| \le 1, \qquad \sum_{i=1}^{t} \lambda_i \mathbf{1}_{J_i} = \tfrac{1}{3}\mathbf{1}_E. \tag{32}$$

The coefficients λ_i define the distribution: choose index i with probability λ_i. Deleting $J_i \cap R_{\text{MS}}$ and doubling $J_i \setminus R_{\text{MS}}$ gives a tour T_i. Consequently, writing $m := |E|$,

$$\sum_i \lambda_i |E(T_i)| = m - \tfrac{1}{3}|R_{\text{MS}}| + \tfrac{1}{3}|E \setminus R_{\text{MS}}| = \tfrac{4}{3}m - \tfrac{2}{3}|R_{\text{MS}}|. \tag{33}$$

At least one of these tours has length at most this bound.

E.2. **The circulation construction.** Suppose that G is not a cycle, and fix a DFS tree T rooted at a vertex r with $\deg_G(r) \ge 3$.

We use the circulation network described in [2, Section 12.4]. Orient the tree edges away from r and subdivide each arc uv by a vertex ζ_{uv}, giving the arcs (u, ζ_{uv}) and (ζ_{uv}, v). Replace each back edge $uw \in I(u, v)$ by the arc (w, ζ_{uv}).

Set the lower bound to one on (u, ζ_{uv}) when $u \ne r$, and to zero on every other arc. A feasible circulation assigns nonnegative flows satisfying these lower bounds and flow conservation.

For each $uv \in E(T)$, write

$$I^{\text{flow}}_{uv} := \{(w, \zeta_{uv}) : uw \in I(u, v)\}.$$

The first unit entering ζ_{uv} through these back arcs is free, and each additional unit has cost one. Thus the cost of a feasible circulation f is

$$C(f) := \sum_{uv \in E(T)} \max\{f(I^{\text{flow}}_{uv}) - 1, 0\}. \tag{34}$$

Figure 11 shows how this cost is linearized.



(A) A nonroot group in the flow network. (B) Linearizing the zero-cost first-unit rule.

FIGURE 11. The local circulation construction of Section 12.4 of [2]. At the root, the first tree arc has lower bound zero.

For an integral feasible circulation f, let

$$E_f := E(T) \cup \{e \in E \setminus E(T) : f_e > 0\}, \qquad G_f := (V, E_f).$$

By [2, Lemma 12.8], G_f is 2-vertex-connected. For every nonroot group with a retained back edge, pair uv with one such edge. Leave the root tree edge unpaired, as in the book. Let $\mathcal{P}_f$ be these pairs, and let R_f consist of all retained back edges and all paired tree edges. Define

$$\Lambda_{\text{MS}}(f) := \tfrac{4}{3}|E_f| - \tfrac{2}{3}|R_f|. \tag{35}$$

For this support and fixed MS removable pairing, let

$$\mathrm{MS}_{\mathrm{OPT}}(f) := \min\left\{ |E_f| - |J \cap R_f| + |J \setminus R_f| : \begin{array}{ll} J \subseteq E_f, & \mathrm{odd}(J) = \mathrm{odd}(G_f), \\ |J \cap P| \le 1 & (P \in \mathcal{P}_f) \end{array} \right\}. \tag{36}$$

Thus $\mathrm{MS}_{\mathrm{OPT}}(f)$ is the minimum tour length obtained by optimizing the parity correction for this support and pairing. The one-third point in Lemma E.1 gives the bound $\Lambda_{\mathrm{MS}}(f)$. Optimizing the parity correction and applying [2, Lemma 12.8] therefore gives

$$\mathrm{MS}_{\mathrm{OPT}}(f) \le \Lambda_{\mathrm{MS}}(f) \le \frac{4}{3}n + \frac{2}{3}C(f) - \frac{2}{3}, \tag{37}$$

where $n := |V|$.

Appendix F. From the circulation bound to the StarDeletionLP

For a fixed DFS tree, we map every feasible Mömke–Svensson circulation to a point of $\mathcal{P}^{\mathrm{SDL}}$ with no larger tour bound. We prove this first for integral circulations; the general case then follows from circulation integrality.

F.1. **Restoring missing pairs.** An integral Mömke–Svensson circulation may omit every back edge in $I(u, v)$, even when this set is nonempty. For a nonroot tree edge uv, restoring one such edge creates an MS removable pair without changing the support bound. The next lemma gives this construction formally.

Lemma F.1 (Support restoration)**.** *Let $G = (V, E)$ be a simple 2-vertex-connected graph that is not a cycle, and let T be a DFS tree rooted at r with $\deg_G(r) \ge 3$. Let f be an integral feasible circulation, with support $G_f = (V, E_f)$ and* MS *removable pairing $(R_f, \mathcal{P}_f)$ as in Section E.2.*

We can restore some of the omitted back edges so that every tree edge uv with $I(u, v) \neq \varnothing$ is paired with an edge of $I(u, v)$ in the resulting support. Write $G_f^+ = (V, E_f^+)$ for this support and $(R_f^+, \mathcal{P}_f^+)$ for its MS *removable pairing. Then $E_f \subseteq E_f^+ \subseteq E$, the DFS tree remains T, and R_f^+ consists of all back edges in E_f^+ together with the paired tree edges. Moreover, these changes do not increase the bound, that is:*

$$\frac{4}{3}|E_f^+| - \frac{2}{3}|R_f^+| \le \Lambda_{\mathrm{MS}}(f). \tag{38}$$

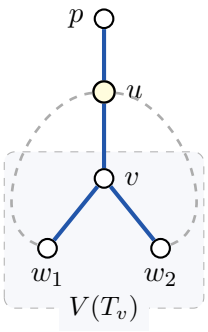
Proof. For each nonroot tree edge uv with $I(u, v) \neq \varnothing$, proceed as follows. If an edge of $I(u, v)$ is retained, keep the existing pair containing uv. Otherwise, restore any $uw \in I(u, v)$ and add the pair $\{uv, uw\}$. This adds one support edge and two removable edges, so the change in the support bound is

$$\frac{4}{3} - \frac{2}{3} \cdot 2 = 0.$$

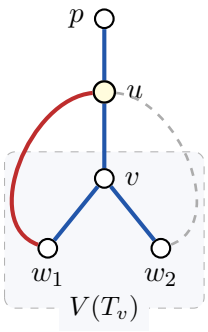
Figure 12 illustrates this operation.

At the root, the tree edge is initially unpaired, while the retained back edges are already removable [2, proof of Lemma 12.8]. The support G_f is 2-vertex-connected, so at least one root back edge is retained. Since $\deg_G(r) \ge 3$, restore a second root back edge if necessary, then pair the root tree edge with a retained root back edge. Restoring an edge and adding this pair leaves the support bound unchanged; adding the pair without restoring an edge decreases it by 2/3. This proves (38).

The pairs are edge-disjoint and their centers have degree at least three. Removability follows from Lemma D.1, applied to T together with one back edge per pair. ∎



(A) No back edge of $I(u, v)$ is retained.



(B) Restore uw_1 and pair it with uv.

FIGURE 12. A nonroot restoration. Gray dashed edges are absent from the support; the tree edge uv is retained in both panels. Restoring uw_1 adds one support edge and two removable edges, so the support bound changes by $\frac{4}{3} - \frac{2}{3} \cdot 2 = 0$. The DFS tree is unchanged.

F.2. **Comparison with the circulation bound.** To compare the StarDeletionLP with the circulation bound, we first lift joins that leave an edge of every star unselected.

Lemma F.2 (Lifting a join that leaves an edge of every star unselected)**.** *Every* $\mathrm{odd}(G)$*-join* $J \subseteq E$ *with* $S \not\subseteq J$ *for all* $S \in \mathcal{S}$ *has a unique lift* $K \subseteq A^{\mathrm{star}}$ *satisfying* $\mathbf{1}_K \in \mathcal{P}^{\mathrm{SDL}}$*. Its representatives are those of* J*, and its connector choices are determined by*

$$\mathbf{1}_{\{c_S \in K\}} \equiv k_S + 1 - |J \cap S| \pmod 2.$$

Proof.

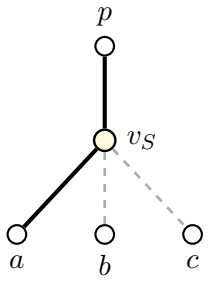
For each $S \in \mathcal{S}$, put $m_S := |J \cap S|$ and choose $\gamma_S \in \{0, 1\}$ with $\gamma_S \equiv k_S + 1 - m_S \pmod 2$. Define K by selecting the representative of every edge of J and selecting c_S exactly when $\gamma_S = 1$. Figure 13 illustrates this construction.

At z_S, the selected degree is $m_S + \gamma_S$, whose parity is $k_S + 1 = \deg_{H^{\mathrm{star}}}(z_S)$. Since $m_S \le k_S - 1$, this degree is at most k_S; its parity therefore makes it at most $k_S - 1$, as required by the auxiliary degree bound.

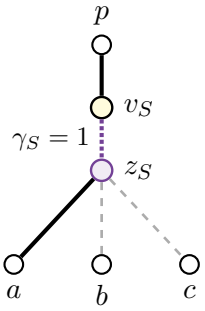
When passing from J in G to K in H^{star}, each star centered at v removes m_S selected incidences at v and adds γ_S selected connector incidences. Hence

$$\deg_K(v) = \deg_J(v) + \sum_{S : v_S = v} (\gamma_S - m_S) \equiv \deg_G(v) + \sum_{S : v_S = v} (1 - k_S) = \deg_{H^{\mathrm{star}}}(v) \pmod 2.$$

Here we used that J is an $\mathrm{odd}(G)$-join and $\gamma_S - m_S \equiv 1 - k_S \pmod 2$. Thus K is an $\mathrm{odd}(H^{\mathrm{star}})$-join. Its degree at each original vertex is at most $\deg_{H^{\mathrm{star}}}(v) = b_v^{\mathrm{star}}$ because $K \subseteq A^{\mathrm{star}}$. Consequently $\mathbf{1}_K \in \mathcal{P}^{\mathrm{SDL}}$. Finally, the parity at z_S forces γ_S, so the lift is unique. ∎



(A) One of the three star edges is selected.



(B) The connector is selected to give even degree at z_S.

FIGURE 13. A local example of the lift with $k_S = 3$ and $m_S = 1$. Black edges are selected, gray dashed edges are unselected, and the purple connector is selected. The selected degree at z_S is $2 = k_S - 1$. The selected degree at v_S remains even.

Theorem F.3 (Comparison with the Mömke–Svensson bound)**.** *Let* $G = (V, E)$ *be a simple* 2*-vertex-connected graph, and fix a DFS tree* T *rooted at* r *with* $\deg_G(r) \ge 3$*. Let* $\mathcal{S}$ *be the* Star *removable system constructed from* T *as in Section D, and put* $n := |V|$ *and* $m := |E|$*.*

For every feasible Mömke–Svensson circulation f, there exist an integral feasible circulation $\widehat{f}$ with $C(\widehat{f}) \le C(f)$ and a point $x \in \mathcal{P}^{\mathrm{SDL}}(G,\mathcal{S})$ such that

$$\begin{aligned}\mathrm{tsp}(G) \le \textsc{StarDeletionLP}_{\mathrm{OPT}}(G,\mathcal{S}) &\le \Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(x)\\ &\le \Lambda_{\mathrm{MS}}(\widehat{f}) = \frac{4}{3}|E_{\widehat{f}}| - \frac{2}{3}|R_{\widehat{f}}|\\ &\le \frac{4}{3}n + \frac{2}{3}C(f) - \frac{2}{3}.\end{aligned}$$

Proof. By circulation integrality [11], we can choose an integral feasible circulation $\widehat{f}$ with $C(\widehat{f}) \le C(f)$.

Apply Lemma F.1 to obtain the support $G^+_{\widehat{f}} = (V, E^+_{\widehat{f}})$ and its MS removable pairing $(R^+_{\widehat{f}}, \mathcal{P}^+_{\widehat{f}})$. Every star of $\mathcal{S}$ now contains a retained pair.

Lemma E.1 gives a convex combination of $\mathrm{odd}(G^+_{\widehat{f}})$-joins J_i in which each retained edge is selected with probability $1/3$ and no J_i contains both edges of a pair. We extend each J_i to an $\mathrm{odd}(G)$-join by adding every edge omitted from $G^+_{\widehat{f}}$. Put

$$D := E \setminus E^+_{\widehat{f}}, \qquad \widetilde{J}_i := J_i \cup D.$$

Indeed,

$$\mathrm{odd}(\widetilde{J}_i) = \mathrm{odd}(G^+_{\widehat{f}}) \triangle \mathrm{odd}(D) = \mathrm{odd}(G).$$

Each star contains a retained pair, and J_i leaves at least one edge of that pair unselected. Since this edge is not in D, $\widetilde{J}_i$ also leaves at least one edge of every star unselected. By Lemma F.2, each $\widetilde{J}_i$ gives an integral point $z_i \in \mathcal{P}^{\mathrm{SDL}}(G,\mathcal{S})$ satisfying

$$(z_i)_{\mathrm{rep}(e)} = \mathbf{1}_{\{e \in \widetilde{J}_i\}} \qquad (e \in E).$$

Let λ_i be the weight of J_i in the convex combination given by Lemma E.1. Using the same weights, define

$$x := \sum_i \lambda_i z_i \in \mathcal{P}^{\mathrm{SDL}}(G,\mathcal{S}).$$

Since $\widetilde{J}_i = J_i \cup D$, we obtain

$$x_{\mathrm{rep}(e)} = \sum_i \lambda_i \mathbf{1}_{\{e \in \widetilde{J}_i\}} = \begin{cases} 1, & e \in D,\\ \frac{1}{3}, & e \in E^+_{\widehat{f}}. \end{cases} \tag{39}$$

We now compute the tour bound given by x. Recall that R and N are the sets of removable and nonremovable edges. Every omitted edge is a back edge and therefore belongs to R. By the construction of $R^+_{\widehat{f}}$,

$$R = D \,\dot\cup\, R^+_{\widehat{f}}, \qquad E^+_{\widehat{f}} = R^+_{\widehat{f}} \,\dot\cup\, N.$$

Substituting the coordinates of x into the objective gives

$$\begin{aligned}\Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(x) &= m - |D| - \frac{1}{3}|R^+_{\widehat{f}}| + \frac{1}{3}|N|\\ &= \frac{4}{3}|E^+_{\widehat{f}}| - \frac{2}{3}|R^+_{\widehat{f}}|\\ &\le \Lambda_{\mathrm{MS}}(\widehat{f}) \qquad (40)\\ &\le \frac{4}{3}n + \frac{2}{3}C(\widehat{f}) - \frac{2}{3}\\ &\le \frac{4}{3}n + \frac{2}{3}C(f) - \frac{2}{3}.\end{aligned}$$

Here the first inequality follows from Lemma F.1, the second from (37), and the last from $C(\widehat{f}) \le C(f)$. The StarDeletionLP optimum is at most $\Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(x)$, so Theorem B.1 gives the claimed tour bound. ∎

Lemma F.4 (Comparison with optimized MS parity correction)**.** *Let G and T satisfy the assumptions of Theorem F.3, and let $\mathcal{S}$ be the* Star *removable system constructed from T as in Section D. For every integral feasible MS circulation f,*

$$\textsc{StarReconnectionLP}_{\mathrm{OPT}}(G,\mathcal{S}) \le \mathrm{MS}_{\mathrm{OPT}}(f).$$

Proof. Let $J \subseteq E_f$ attain $\mathrm{MS}_{\mathrm{OPT}}(f)$, and put

$$D := E \setminus E_f, \qquad U := (R \cap E_f) \setminus R_f, \qquad \widetilde{J} := J \cup D.$$

All edges of D are back edges and hence belong to R. Also, $R_f \subseteq R$, and U consists of the tree edges that belong to our stars but are unpaired in the MS construction. We have

$$\mathrm{odd}(\widetilde{J}) = \mathrm{odd}(G_f) \triangle \mathrm{odd}(D) = \mathrm{odd}(G).$$

If $\widetilde{J}$ contains an entire star, its tree edge cannot be paired in $\mathcal{P}_f$, since J leaves at least one edge of every pair in $\mathcal{P}_f$ unselected. Its tree edge therefore belongs to $J \cap U$. Distinct stars have distinct tree edges, so

$$|\{S \in \mathcal{S} : S \subseteq \widetilde{J}\}| \le |J \cap U|.$$

Since $\widetilde{J}$ is an $\mathrm{odd}(G)$-join, the parity argument in the proof of Lemma F.2 extends it to an $\mathrm{odd}(H^\circ)$-join K satisfying

$$\mathrm{rep}(e) \in K \quad \Longleftrightarrow \quad e \in \widetilde{J} \qquad (e \in E).$$

This argument also applies when $\widetilde{J}$ contains an entire star. Put $y := \mathbf{1}_K \in \mathcal{P}^\circ$.

At z_S, the selected degree has the same parity as $k_S + 1$. It equals $k_S + 1$ when $S \subseteq \widetilde{J}$ and is at most $k_S - 1$ otherwise. Thus (27) gives

$$\begin{aligned}
\Phi_{G,\mathcal{S}}(y) &= m - |\widetilde{J} \cap R| + |\widetilde{J} \cap N| + 2|\{S \in \mathcal{S} : S \subseteq \widetilde{J}\}| \\
&= \mathrm{MS}_{\mathrm{OPT}}(f) - 2|J \cap U| + 2|\{S \in \mathcal{S} : S \subseteq \widetilde{J}\}| \\
&\le \mathrm{MS}_{\mathrm{OPT}}(f).
\end{aligned}$$

Minimizing over $\mathcal{P}^\circ$ proves the claim. ∎

Corollary F.5 (Comparison for a fixed DFS tree)**.** *Let G be a simple 2-vertex-connected graph, and fix a DFS tree T rooted at r with $\deg_G(r) \ge 3$. Let $\mathcal{S}$ be the* Star *removable system constructed from T as in Section D. Then*

$$\begin{aligned}
\mathrm{tsp}(G) &\le \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G,\mathcal{S}) \\
&\le \min\left\{ \textsc{StarDeletionLP}_{\mathrm{OPT}}(G,\mathcal{S}), \min_f \mathrm{MS}_{\mathrm{OPT}}(f) \right\} \\
&\le \frac{4}{3}n + \frac{2}{3}\min_f C(f) - \frac{2}{3}.
\end{aligned}$$

Here the minima are over minimum-cost integral MS circulations for T, and $\mathrm{MS}_{\mathrm{OPT}}(f)$ is defined in (36). The comparison follows from Theorem F.3, Lemma F.4, and Proposition C.3. Thus the StarDeletionLP *gives a tour bound at least as strong as the optimized MS circulation bound, and the* StarReconnectionLP *also bounds the optimized MS parity correction.*

Appendix G. A 13/10 bound for 4-regular, 4-connected graphs

Assigning edge values in $[1/3, 2/3]$ makes the Blossom inequalities easy to satisfy on cuts with at least three edges, since each edge contributes at least $1/3$. Smaller cuts must be checked separately. Here we show that higher connectivity allows us to use other edge values while still obtaining a point in $\mathcal{P}^\circ(G, \mathcal{S})$. For 4-regular, 4-vertex-connected graphs, this leads to the improved approximation bound of 13/10.

We start with an initial vector x^{init} for which the objective expression in (41) equals $5n/4 - 1/2$. The initial vector need not belong to $\mathcal{P}^\circ$. We identify the Blossom cuts whose inequalities are not yet satisfied and increase the value of an edge in each such cut so that its inequality becomes satisfied. Some of these changes leave the StarReconnectionLP objective unchanged, while others increase it by $1/4$. We prove that at most $(n-2)/5$ changes that increase the objective are needed, and then verify that the resulting vector $\bar{x}$ belongs to $\mathcal{P}^\circ$. This gives a tour of length at most $13n/10 - 3/5$.

The existing 4/3 baseline. For this graph class, a 4/3 bound already follows from Newman's subquartic result [5, Theorem 1]. Indeed, for the graph subtour relaxation

$$\text{SubtourLP}(G) := \min\{y(E) : y \geq 0,\ y(\delta_G(U)) \geq 2 \text{ for } \varnothing \neq U \subsetneq V\},$$

the vector $y_e := 1/2$ is feasible because every cut has at least four edges. Since $m = 2n$, its objective is n, and summing the singleton-cut constraints gives the matching lower bound $\text{SubtourLP}(G) \geq n$. Hence $\text{SubtourLP}(G) = n$, which is exactly the case covered by that theorem.

Throughout this section, $G = (V, E)$ is simple, 4-regular, and 4-vertex-connected, and $n := |V|$. Fix an arbitrary DFS tree T rooted at r. We use the StarReconnectionLP with the Star removable system $\mathcal{S}$ constructed from T as in Section D. We abbreviate $\mathcal{P}^\circ(G, \mathcal{S})$ by $\mathcal{P}^\circ$.

In the DFS diagrams in this section, blue edges represent edges of T and red curves represent back edges; the same colors are used for their representatives. The DFS tree is drawn downward from r, so its edges are read from parent to child and back edges from descendant to ancestor. Dotted purple edges are connectors.

G.1. Vertex types and the Star removable system.

For a nonroot vertex v, let $j(v)$ be its number of DFS children and let

$$i(v) := \big|\{vw \in E \setminus E(T) : w \text{ is a proper descendant of } v\}\big|.$$

Thus $i(v)$ counts the back edges whose upper endpoint is v; we call these the back edges entering v. The graphs and their back edges remain undirected. Define

$$B_2 := \{v \in V \setminus \{r\} : j(v) = 2,\ i(v) = 1\}.$$

For any nonroot vertex u, its parent edge is $\text{par}(u)u$. A vertex $u \in B_2$ has one parent edge, two child edges, and one back edge entering u, with no back edge from u to a proper ancestor. For consistency with the earlier sections, we recall the notation. For each tree edge uv, let $I(u, v)$ be the set of back edges joining u to vertices in the subtree rooted at v. If $I(u, v) \neq \varnothing$, we form the star

$$S(u, v) = \{uv\} \cup I(u, v).$$

Write

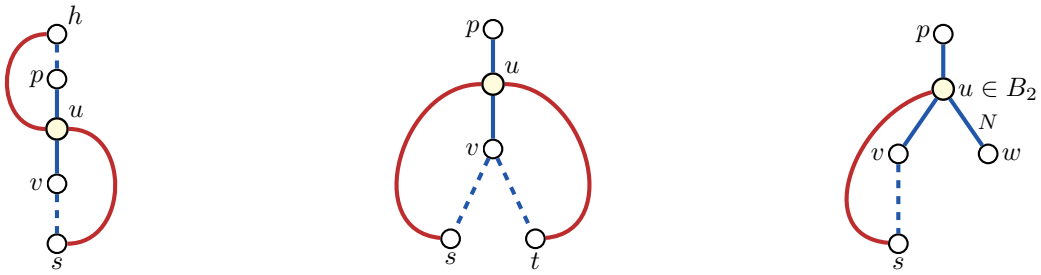
$$R = \bigcup_{S \in \mathcal{S}} S, \qquad N = E(G) \setminus R.$$

Thus R consists of the removable edges, and N consists of the remaining edges, which are also called nonremovable edges.

Let v be a nonroot vertex with only one DFS child w. Since G is 4-regular and v already has a parent edge and the child edge vw, at most two back edges can end at v. If none ends at v, no star

is formed there and $vw \in N$. If one back edge ends at v, it forms a star of size two together with vw. If two back edges end at v, they form a star of size three together with vw.

A nonroot vertex with two DFS children either belongs to B_2, in which case one child edge belongs to a star of size two and the other belongs to N, or has both child edges in N. A nonroot vertex with three DFS children has all three child edges in N. Finally, the root has exactly one DFS child and three back edges ending at it, which together form a star of size four.



(A) $j(u) = 1$, $i(u) = 1$: a star of size two.

(B) $j(u) = 1$, $i(u) = 2$: one triple.

(C) $u \in B_2$: uv belongs to a star of size two and $uw \in N$.

FIGURE 14. Nonroot vertices with nonempty stars. Blue edges belong to the DFS tree; red curves are back edges. Dashed blue segments denote tree paths. In (c), the back edge ending at u comes from the subtree rooted at v, and $uw \in N$.

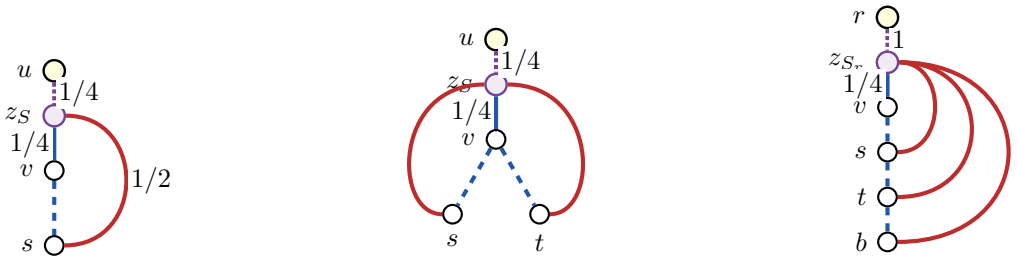
Theorem G.1. *For every fixed DFS tree T of a simple, 4-regular, 4-vertex-connected graph G on n vertices, the* StarReconnectionLP *with the* Star *removable system $\mathcal{S}$ constructed from T as in Section D certifies*

$$\mathrm{tsp}(G) \le \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}) \le \frac{13}{10}n - \frac{3}{5}.$$

A vector in $\mathcal{P}^\circ$ attaining this bound can be constructed in polynomial time.

G.2. **The initial vector and its objective.** Define the initial vector $x^{\mathrm{init}} \in \mathbb{R}^{A^\circ}$ on H°. Set $x^{\mathrm{init}}_e := 1/4$ for every $e \in N_H$ and use the following values on the stars:

S	$\mathrm{rep}(e)$ $(e \in S \cap E(T))$	$\mathrm{rep}(b)$ $(b \in S \setminus E(T))$	$x^{\mathrm{init}}_{c_S}$
nonroot star of size two	1/4	1/2	1/4
nonroot triple	1/4	3/4	1/4
root star	1/4	3/4	1



(A) A nonroot star of size two. (B) A nonroot triple; red edges have value 3/4. (C) The root star; red edges have value 3/4.

FIGURE 15. Coordinates of x^{init} on H°. Blue edges come from tree edges, curved red edges from back edges, and dotted purple edges are connectors. Dashed blue segments denote tree paths with omitted intermediate vertices. Every edge in N_H has value 1/4.

Each star S contains $k_S - 1$ back edges, and every back edge belongs to exactly one star. Since there are $m - n + 1$ back edges, the constant in (27) is

$$m - \sum_{S \in \mathcal{S}} (k_S - 1) = m - (m - n + 1) = n - 1.$$

For the values just assigned, every star satisfies

$$k_S - 1 - x^{\text{init}}(S_H) \le x^{\text{init}}_{c_S}.$$

Thus the maximum in the initial objective equals $x^{\text{init}}_{c_S}$. Let $\bar{x}$ denote the vector obtained from x^{init} by the correction procedure in Section G.3. Since this procedure only increases coordinates, every star satisfies $k_S - 1 - \bar{x}(S_H) \le \bar{x}_{c_S}$. Its objective expression is

$$\begin{aligned} &n - 1 + \bar{x}(N_H) + \sum_{S \in \mathcal{S}} \max\{\bar{x}_{c_S},\, k_S - 1 - \bar{x}(S_H)\} \\ &\quad = n - 1 + \bar{x}(N_H) + \sum_{S \in \mathcal{S}} \bar{x}_{c_S}. \end{aligned} \tag{41}$$

Increasing the coordinate of $\text{rep}(e)$ for $e \in R$ leaves the objective unchanged. Increasing a connector coordinate or the coordinate of $\text{rep}(e)$ for $e \in N$ by 1/4 increases it by 1/4.

In x^{init}, every edge of N_H and every nonroot connector has value 1/4, while the root connector has value one. Using $|N| + |\mathcal{S}| = n - 1$, the initial objective expression is

$$\begin{aligned} n - 1 + x^{\text{init}}(N_H) + \sum_{S \in \mathcal{S}} x^{\text{init}}_{c_S} &= n - 1 + \frac{1}{4}|N| + \frac{1}{4}(|\mathcal{S}| - 1) + 1 \\ &= n - 1 + \frac{1}{4}(n - 2) + 1 \\ &= \frac{5}{4}n - \frac{1}{2}. \end{aligned} \tag{42}$$

It remains to satisfy the Blossom inequalities while bounding the number of increases that raise this value.

G.3. **The feasibility correction procedure.** We show that only two types of cuts in H° need to be corrected: cuts around certain vertices and cuts around certain triangles. In each case, the three boundary edges have value 1/4 in x^{init}. Increasing at least one of them to 1/2 satisfies the Blossom inequalities for that cut.

Objective-preserving parent-edge changes. Each nonroot vertex centers at most one star: two edge-disjoint stars, together with its parent edge, would require at least five incident edges, contradicting 4-regularity. For each vertex v that centers a star of size two, let S_v denote this star. Its auxiliary vertex and connector are z_{S_v} and c_{S_v}.

In G, for $u \in B_2$, let w be the child whose subtree contains no back edge ending at u. The edge uw belongs to N, and the three edges incident with the original copy of u in H° are

$$\delta_{H^\circ}(u) = \{\mathrm{rep}(\mathrm{par}(u)u), \mathrm{rep}(uw), c_{S_u}\}.$$

Initially, all three have value $1/4$, so

$$x^{\mathrm{init}}(\delta_{H^\circ}(u)) = \frac{3}{4} < 1.$$

Thus, the cut inequality $\bar{x}(\delta_{H^\circ}(u)) \geq 1$ requires an increase of at least $1/4$ in this boundary sum.

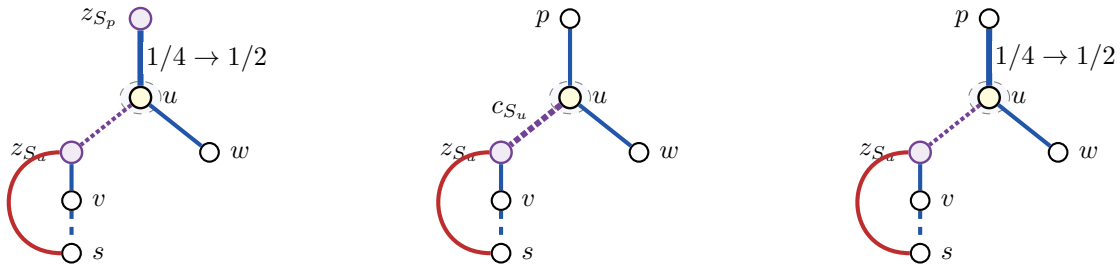
Whenever $\mathrm{par}(u)u \in R$, increase its representative from $1/4$ to $1/2$. This raises the boundary sum to one without changing (41).

The vertices of B_2 whose cut inequalities remain unsatisfied after these changes are precisely those whose parent edges belong to N. Define

$$\mathcal{U} := \{u \in B_2 : \mathrm{par}(u)u \in N\}. \tag{43}$$

For every $u \in \mathcal{U}$, the three incident coordinates are still $1/4$. We call this a **vertex demand**: at least one of these coordinates must be increased. Increasing an edge of N_H by $1/4$ increases (41) by $1/4$.

Figure 16 illustrates the objective-preserving change and the two types of changes that increase (41).



(A) $pu \in R$: cost 0. (B) Increase c_{S_u}: cost $1/4$. (C) $pu \in N$: cost $1/4$.

FIGURE 16. Three ways to satisfy the cut around $u \in B_2$. In each panel, increase the thick edge from $1/4$ to $1/2$. The shaded set is $\{u\}$. In (a), the parent edge belongs to a star centered at p; the same change applies if that star has size three or is the root star.

Triangle demands. The initial vector x^{init} can also violate Blossom inequalities for cuts separating a triangle from the rest of the graph. These cuts also require a coordinate increase.

In G, consider a vertex $u \in B_2$ and put $p := \mathrm{par}(u)$. Let v be the child whose subtree contains the lower endpoint s of the unique back edge su ending at u, and let w be the other child. Suppose that $\{pu, pw\}$ is the star centered at p, so pw is a back edge with upper endpoint p and lower endpoint w. Suppose also that w has exactly one DFS child a and exactly one back edge tw entering it, with $t \in V(T_a)$. The back edges pw and tw are distinct: the first joins w to an ancestor, and the second joins it to a descendant. The three stars are

$$\{pu, pw\}, \qquad \{uv, su\}, \qquad \{wa, tw\},$$

centered at p, u, w, respectively.

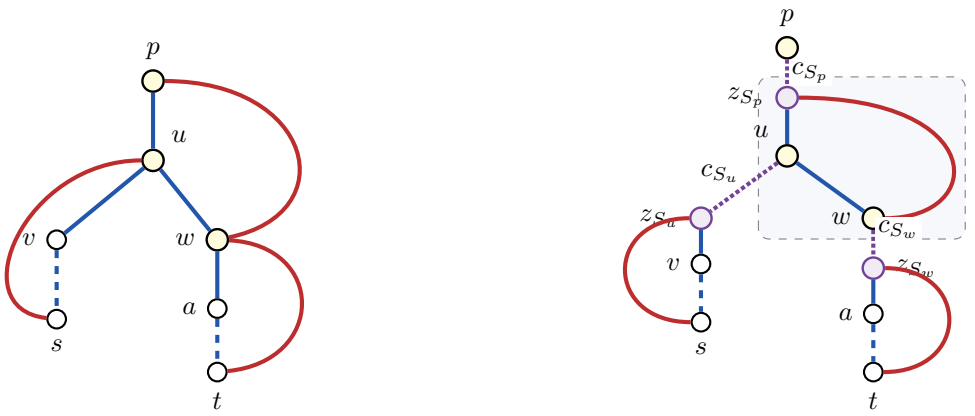
In H°, the vertices

$$X_u := \{z_{S_p}, u, w\}$$

form a triangle, with

$$E(H^\circ[X_u]) = \{\mathrm{rep}(pu), \mathrm{rep}(uw), \mathrm{rep}(wp)\} = \{z_{S_p}u, uw, wz_{S_p}\},$$
$$\delta_{H^\circ}(X_u) = \{c_{S_p}, c_{S_u}, c_{S_w}\}.$$

The representatives of wa and tw are incident with z_{S_w}. The connector c_{S_w} joins the original copy of w to z_{S_w}. Figure 17 shows the construction. Let $\mathcal{D}$ be the set of vertices $u \in B_2$ for which this configuration occurs. Since $pu \in R$, every such u lies outside $\mathcal{U}$.



(A) In G: the stars are $\{pu, pw\}$, $\{uv, us\}$, and $\{wa, wt\}$.

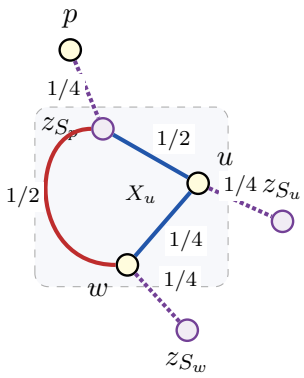
(B) In H°: the three connectors form the boundary.

FIGURE 17. A triangle demand at u. The shaded vertices in H° are $X_u = \{z_{S_p}, u, w\}$; its boundary consists of the three connectors $c_{S_p}, c_{S_u}, c_{S_w}$. Dashed blue segments denote tree paths.

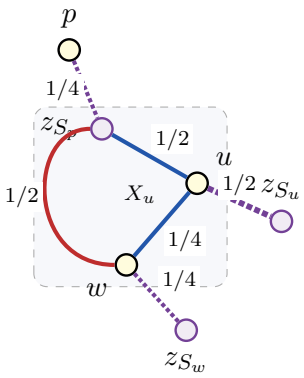
The objective-preserving change has already increased the coordinate of $\mathrm{rep}(pu)$ to $1/2$. This fixes the cut around u, but $\mathrm{rep}(pu)$ is internal to X_u and does not affect its boundary sum. The three boundary connectors retain their initial values, so this sum remains

$$x^{\mathrm{init}}(\delta_{H^\circ}(X_u)) = x^{\mathrm{init}}_{c_{S_p}} + x^{\mathrm{init}}_{c_{S_u}} + x^{\mathrm{init}}_{c_{S_w}} = \frac{3}{4} < 1.$$

We call this the triangle demand at u. To meet it, increase any one of these three connector coordinates from $1/4$ to $1/2$. The boundary sum then becomes one. Figure 18 contrasts the internal parent-edge change with a boundary correction.



(A) Increasing rep(pu) leaves the boundary sum at 3/4.



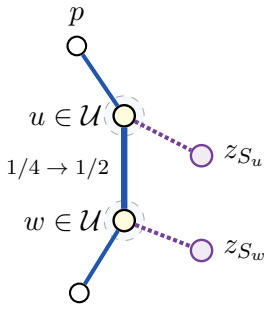
(B) Connector increased: boundary sum 1.

FIGURE 18. An internal change does not satisfy the triangle-cut inequality. On the right, increasing the value on c_{S_u} by 1/4 meets the triangle demand and increases (41) by 1/4. The same boundary correction can be made using c_{S_p} or c_{S_w}.

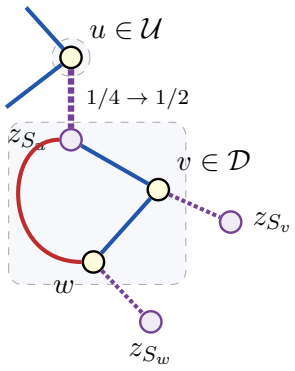
Satisfying two demands with one increase. Each remaining correction increases a connector or an edge of N_H from 1/4 to 1/2. By (41), this increases the objective by 1/4; we call it a *paid increase*.

We can reduce the number of paid increases by choosing edges that belong to the boundaries of two demands. For example, suppose that $u, w \in \mathcal{U}$, where w is a child of u and $uw \in N$. Increasing rep(uw) satisfies both vertex demands, with a total objective increase of 1/4. A connector can also belong to two demand boundaries, as shown in Figure 19.

We next construct a graph recording these shared edges and use a matching to choose the corrections.



(A) Increasing rep(uw), where $uw \in N$, satisfies both vertex demands.



(B) The connector c_{S_u} meets the vertex demand at u and the triangle demand at $v \in \mathcal{D}$.

FIGURE 19. A single increase from 1/4 to 1/2 satisfies both shaded cuts. The thick edge is shared by two vertex demands on the left, and by a vertex demand and a triangle demand on the right.

Selecting the paid increases. For each demand u, let $\mathcal{E}_u \subseteq E(H^\circ)$ be its three boundary edges. More explicitly, put $p := \mathrm{par}(u)$ and let w be the child whose subtree contains no back edge ending

at u. Then

$$\mathcal{E}_u = \{\mathrm{rep}(pu), c_{S_u}, \mathrm{rep}(uw)\}, \quad u \in \mathcal{U},$$
$$\mathcal{E}_u = \{c_{S_p}, c_{S_u}, c_{S_w}\}, \quad u \in \mathcal{D}.$$

Construct a graph K whose vertices are the demands in $\mathcal{U} \dot\cup \mathcal{D}$. Join two distinct demands when their boundary sets share an edge. An edge of K therefore records an increase that satisfies both demands.

Let M be a maximum matching in K. For each $uu' \in M$, we satisfy both demands by choosing an edge in $\mathcal{E}_u \cap \mathcal{E}_{u'}$ and increasing its value from $1/4$ to $1/2$. For every demand u not covered by M, we increase c_{S_u} to $1/2$. If an edge is chosen more than once, we increase its value only once. Let $\bar{x}$ denote the vector obtained from x^{init} after all objective-preserving and paid increases.

An edge of N_H is increased only when it belongs to the boundaries of two matched vertex demands. Both endpoints of such an edge must therefore lie in $\mathcal{U}$.

Now consider a nonroot vertex v of G with one DFS child. Since every vertex in $\mathcal{U}$ has two DFS children, $v \notin \mathcal{U}$. Its parent edge cannot therefore join two vertex demands and is not chosen for such an increase. All other paid increases are on connectors, so the value on $\mathrm{rep}(\mathrm{par}(v)v)$ remains unchanged. We will use this observation later when verifying the Blossom inequalities.

We use one increase for each edge of M and at most one for each of the $|V(K)| - 2|M|$ unmatched demands. Thus the number of paid increases is at most

$$\rho := |M| + \big(|V(K)| - 2|M|\big) = |V(K)| - \nu(K), \tag{44}$$

where $\nu(K) := |M|$ is the maximum matching size. We first prove that $\bar{x} \in \mathcal{P}^\circ$. We then show that K is a forest and use its structure to prove $\rho \le (n-2)/5$.

G.4. **Feasibility of the final vector.** Let $\bar{x}$ be the vector obtained from x^{init} by the corrections in Section G.3. Every coordinate of $\bar{x}$ belongs to $\{1/4, 1/2, 3/4\}$, except $\bar{x}_{c_{S_r}} = 1$. To prove $\bar{x} \in \mathcal{P}^\circ$, it remains to verify the Blossom inequalities in (26):

$$\bar{x}\big(\delta_{H^\circ}(X) \setminus F\big) + |F| - \bar{x}(F) \ge 1 \quad \text{for every } X \subseteq W, \tag{45}$$
$$F \subseteq \delta_{H^\circ}(X) \text{ with } |\delta_{H^\circ}(X)| + |F| \text{ odd}.$$

At every degree-two vertex of H°, both incident edges have value $1/4$. Suppress each such vertex by replacing its two incident edges with a single edge. Also delete the root connector and its degree-one endpoint r. Denote the resulting graph by Q, and define $\bar{x}^Q \in \mathbb{R}^{E(Q)}$ by assigning value $1/4$ to each replacement edge and retaining $\bar{x}_a$ on every unchanged edge a. See Figure 20. We also write $\mathrm{rep}(e)$ for the corresponding edge in Q, with each replacement edge inheriting the label of the parent edge in the path it replaces. We use Q and $\bar{x}^Q$ only to verify the Blossom inequalities.



(A) In H°, the original vertex v has degree two.

(B) In Q, v is suppressed and z_S has degree four.

FIGURE 20. Suppressing a degree-two vertex preserves the common value $1/4$ on the replacement edge. Here $\bar{p}$ is the other endpoint of the parent edge in H°. The root connector, of value one, is deleted together with its degree-one endpoint. Dashed blue segments denote tree paths.

Proposition G.2 (An equivalent cut verification)**.** *The vector $\bar{x}$ belongs to $\mathcal{P}^\circ$ if and only if $\bar{x}^Q$ belongs to the* odd(Q)*-join polytope. Moreover, Q is simple and every vertex has degree three or four. Its degree-three vertices are the auxiliary vertices of stars of size two, the vertices in B_2, and the vertices of G with one DFS child and one back edge ending at them.*

Proof. For a suppressed vertex, crossing one incident edge gives the same contribution as the replacement edge. If both edges cross and exactly one belongs to F, their contribution is $1/4+3/4 = 1$. Otherwise, removing both preserves the parity condition and only decreases the left-hand side.

A crossing root connector contributes one if it is outside F, satisfying the inequality, and zero otherwise. In the latter case, deleting it preserves the parity condition. Conversely, every reduced cut inequality lifts unchanged.

After relabeling the surviving auxiliary vertices of the root star and stars of size three by their centers, Q is obtained from G by applying the construction of H° only to the stars of size two. This preserves simplicity and gives the stated degrees and vertex types. ■

G.5. **Verifying the small cuts.** Every coordinate of $\bar{x}^Q$ lies in $[1/4, 3/4]$. Hence, for $C := \delta_Q(X)$ and $F \subseteq C$,

$$\bar{x}^Q(C \setminus F) + |F| - \bar{x}^Q(F) \geq \frac{1}{4}|C| \geq 1 \qquad \text{if } |C| \geq 4. \tag{46}$$

It remains to consider cuts with at most three edges. The next lemma shows that these cuts isolate a vertex or a triangle.

Lemma G.3 (Cuts with at most three edges)**.** *The graph Q has no bridge or two-edge cut. Every three-edge cut has one of the following forms:*

$$\begin{cases} \delta_Q(\{u\}), & \deg_Q(u) = 3, \\ \delta_Q(\{u,v,w\}), & uv, vw, uw \in E(Q), \quad \deg_Q(u) = \deg_Q(v) = \deg_Q(w) = 3. \end{cases}$$

Proof. Let $C := \delta_Q(X)$ have at most three edges, with both shores nonempty, and let s be the number of connectors in C. For each connector $c_S = uz_S \in C$, delete the vertex u from G. These are s distinct vertices. Also delete the edges of G corresponding to the nonconnector edges of C, namely

$$\{e \in E(G) : \mathrm{rep}(e) \in C\}.$$

Denote the resulting graph by G_C.

After the vertex deletions, the graph is $(4-s)$-vertex-connected and hence $(4-s)$-edge-connected. We then delete at most $3-s$ additional edges, so G_C is connected.

Now, in Q, delete both u and z_S for each connector $c_S = uz_S \in C$, together with the nonconnector edges of C. Contracting the remaining connectors gives exactly G_C, which we have just shown is connected. Since all edges crossing the cut were deleted and each contracted connector lies within one shore, all remaining vertices must lie on the same side of the cut. Replacing X by its complement if necessary, we may therefore assume that every vertex of X was deleted. Each crossing connector has one endpoint in X, so

$$1 \leq |X| \leq s \leq 3.$$

If $|X| = 1$, its vertex must have degree three. If $|X| = 2$, simplicity gives

$$|\delta_Q(X)| \geq 3 + 3 - 2 = 4,$$

which is impossible. Finally, if $|X| = 3$, then

$$|\delta_Q(X)| = \sum_{v \in X} \deg_Q(v) - 2|E(Q[X])| \geq 9 - 6 = 3.$$

Equality holds only when the three vertices form a triangle and each has degree three. This proves both assertions. ■

Lemma G.4 (Vertex cuts)**.** *The vector $\bar{x}^Q$ satisfies all Blossom inequalities for cuts of the form $\delta_Q(\{u\})$.*

Proof. Cuts around degree-four vertices satisfy (46). For a three-edge cut $C = \{e_1, e_2, e_3\}$, the Blossom parity condition allows $|F| = 0$ or 2, so the inequalities are

$$\begin{aligned} &\bar{x}^Q_{e_1} + \bar{x}^Q_{e_2} + \bar{x}^Q_{e_3} \geq 1, \\ &\bar{x}^Q_{e_i} + \bar{x}^Q_{e_j} - \bar{x}^Q_{e_k} \leq 1 \qquad (\{i,j,k\} = \{1,2,3\}). \end{aligned} \tag{47}$$

Initially, the only unsatisfied vertex cuts are those around $u \in B_2$, where all three incident values are $1/4$. The correction procedure raises at least one of them to $1/2$. Every other degree-three vertex already has an incident value of at least $1/2$. Thus the first inequality holds at every vertex.

At any degree-three vertex, at most one incident value is $3/4$, and the others are at most $1/2$. Since every value is at least $1/4$, the second inequality follows from

$$\frac{3}{4} + \frac{1}{2} - \frac{1}{4} = 1.$$

■

It remains to check the triangle cuts. The next lemma shows that they are exactly the cuts used to define the triangle demands.

Lemma G.5 (Triangle cuts)**.** *Every triangle of degree-three vertices in Q has the form*

$$Q[\{z_{S_p}, u, w\}], \qquad p := \mathrm{par}(u), \quad u \in \mathcal{D},$$

where w is the child of u whose subtree contains no back edge ending at u. Its boundary is $\{c_{S_p}, c_{S_u}, c_{S_w}\}$, and $\bar{x}^Q$ satisfies all Blossom inequalities for this cut.

Proof. A triangle cannot contain a connector: its other two edges would correspond to parallel edges in G. Thus each vertex of the triangle has its connector on the boundary and its other two incident edges inside.

A vertex in B_2 has two incident tree edges. Every other degree-three vertex has one tree edge and one back edge, in addition to its connector. Let b denote the number of vertices of the triangle that belong to B_2. The number of tree edges in the triangle is

$$\frac{2b + (3-b)}{2} = \frac{3+b}{2}.$$

Since this is an integer, $b = 1$ or $b = 3$. The latter would make all three triangle edges tree edges, which is impossible. Hence the triangle contains exactly one vertex $u \in B_2$.

The two triangle edges at u correspond to its parent edge and the edge $uw \in N$, where w is its child. Since $w \notin B_2$ and has degree three in Q, it has exactly one DFS child and one back edge ending at it.

Put $p := \mathrm{par}(u)$. We now show that pu belongs to a star of size two centered at p, so the third vertex of the triangle is z_{S_p}.

Suppose first that $pu \in N$. Then the third vertex is p, which has degree three in Q. By Proposition G.2, either $p \in B_2$ or p centers a star of size two and has exactly one DFS child. If $p \in B_2$, the triangle would contain two vertices of B_2, namely p and u, contradicting $b = 1$. In the latter case, u is the only DFS child of p, so pu belongs to this star, contradicting $pu \in N$.

Thus pu belongs to a star centered at p. This star must have size two, since a star of size three or the root star would give a vertex of degree four in Q. The third triangle vertex is therefore z_{S_p}. The edge joining z_{S_p} to w corresponds to the back edge pw, so

$$S_p = \{pu, pw\}.$$

This is exactly the configuration defining $u \in \mathcal{D}$, shown in Figure 17, with

$$\delta_Q(\{z_{S_p}, u, w\}) = \{c_{S_p}, c_{S_u}, c_{S_w}\}.$$

The correction procedure increases at least one of these connector values from 1/4 to 1/2. All three values belong to $\{1/4, 1/2\}$, so their sum is at least one and the sum of any two is at most one. Thus (47) holds. ■

Proposition G.6 (The final vector)**.** *The vector $\bar{x}$ obtained by the feasibility correction procedure belongs to $\mathcal{P}^\circ$.*

Proof. By Lemma G.3, every cut of Q with at most three edges is a vertex cut or a triangle cut. Their Blossom inequalities hold by Lemmas G.4 and G.5. Larger cuts satisfy (46). Hence $\bar{x}^Q$ belongs to the odd(Q)-join polytope, and Proposition G.2 gives $\bar{x} \in \mathcal{P}^\circ$. ■

G.6. **The structure of shared corrections.** It remains to bound how many paid increases the procedure uses. Recall that K has one vertex for each remaining demand:

$$V(K) = \mathcal{U} \dot\cup \mathcal{D}.$$

Two distinct demands are adjacent when their boundaries share an edge. Formally, for $u, v \in V(K)$ with $u \neq v$,

$$uv \in E(K) \quad \Longleftrightarrow \quad \mathcal{E}_u \cap \mathcal{E}_v \neq \varnothing,$$

where $\mathcal{E}_u$ is the set of boundary edges of the demand at u. Locate a vertex demand at its vertex and a triangle demand at its middle vertex. For adjacent demands, call the one higher in the DFS tree the predecessor and the lower one its successor. The proof below shows that the two locations are always comparable in the DFS tree. We use this structure to bound $\rho = |V(K)| - \nu(K)$.

Lemma G.7 (The correction forest)**.** *The graph K is a forest, and every vertex has at most one predecessor. A vertex in $\mathcal{U}$ has at most one successor in $\mathcal{U}$ and one in $\mathcal{D}$. A vertex in $\mathcal{D}$ has at most two successors, both in $\mathcal{D}$. In particular, $K[\mathcal{U}]$ is a disjoint union of paths.*

Proof. To understand the edges of K, we determine which two demands can share a correction. We show that every edge of K comes from one of the following three cases. We then use these cases to bound the number of predecessors and successors of each demand.

Recall that a triangle demand at u has boundary

$$\{c_{S_p}, c_{S_u}, c_{S_w}\}, \qquad p = \mathrm{par}(u),$$

where w is the child of u whose subtree contains no back edge ending at u. This vertex w has exactly one DFS child.

Case 1: Two demands share a nonremovable tree edge. Let $uw \in N$, where w is a child of u. The edge rep(uw) can belong only to vertex-demand boundaries, since triangle boundaries contain only connectors. The two demands must therefore be at its endpoints, so $u, w \in \mathcal{U}$. Conversely, if $u, w \in \mathcal{U}$, their boundaries both contain rep(uw), giving the edge uw in K. Here w is the child of u whose subtree contains no back edge ending at u.

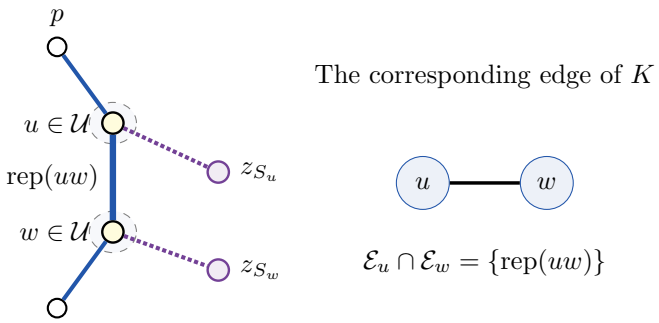


FIGURE 21. Case 1. The nonremovable edge uw joins two vertices in $\mathcal{U}$. Its copy in H° crosses both singleton cuts, so one increase satisfies both vertex demands.

Case 2: Two demands share a connector centered at $u \in B_2$. Let v be the child of u for which $uv \in S_u$. The connector c_{S_u} belongs to the boundary of the vertex demand at u if $u \in \mathcal{U}$, and to the boundary of the triangle demand at u if $u \in \mathcal{D}$. These demands cannot both occur at u: a triangle demand requires $\mathrm{par}(u)u \in R$, whereas $u \in \mathcal{U}$ requires $\mathrm{par}(u)u \in N$.

The connector can also belong to the boundary of the triangle demand at v, when $v \in \mathcal{D}$. Thus it can be shared by a vertex demand and a triangle demand, or by two triangle demands.

There are no other demands whose boundaries contain c_{S_u}. Indeed, a triangle demand whose parent is u must be at v, because its triangle contains the tree edge of S_u, namely uv. For any triangle demand at t, its boundary is

$$\{c_{S_p}, c_{S_t}, c_{S_w}\}, \qquad p := \mathrm{par}(t),$$

where w is a child of t with exactly one DFS child. Since $u \in B_2$ has two DFS children, u cannot be this vertex w.

Thus the only demands whose boundaries can contain c_{S_u} are the demand at u and the triangle demand at v. They share this connector when both demands exist. Therefore, c_{S_u} gives the edge uv in K exactly when

$$u \in \mathcal{U} \cup \mathcal{D} \qquad \text{and} \qquad v \in \mathcal{D}.$$



(A) $u \in \mathcal{U}$, $v \in \mathcal{D}$. (B) $u, v \in \mathcal{D}$.

FIGURE 22. Case 2. Both configurations give the edge uv in K. The shared connector c_{S_u} crosses the two shaded cuts. On the left they are $\{u\}$ and $\{z_{S_u}, v, w\}$; on the right they are $\{z_{S_p}, u, a\}$ and $\{z_{S_u}, v, b\}$. The vertex u has two DFS children in both configurations.

Case 3: Two triangle demands share a connector at a vertex with exactly one DFS child. Let $p := \mathrm{par}(w)$, and let v be the only DFS child of w. Suppose that w is the center of a star S_w of size two. There is no demand at w, since every vertex in $\mathcal{U} \cup \mathcal{D}$ has two DFS children.

The connector c_{S_w} can belong to the boundary of a triangle demand at p, when w is the child of p in that triangle. It can also belong to the boundary of a triangle demand at v, whose parent is w. These are the only possibilities. When both occur, c_{S_w} is shared by the two triangle demands and gives the edge pv in K.

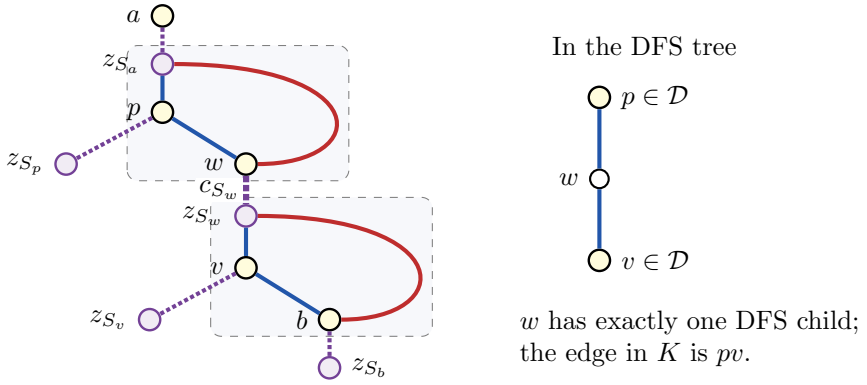


FIGURE 23. Case 3. Here $a = \mathrm{par}(p)$ and $p = \mathrm{par}(w)$. The shared connector c_{S_w} crosses the cuts around $\{z_{S_a}, p, w\}$ and $\{z_{S_w}, v, b\}$. The demands are at p and v; there is no demand at w.

These cases cover every possible shared correction. They also show that adjacent demands are located at vertices where one is a descendant of the other.

For a vertex demand at u, the only possible predecessor comes from its nonremovable parent edge, by Case 1. Its possible successors are a vertex demand at the child joined to u by a nonremovable edge, and a triangle demand at its other child, by Case 2. Thus it has at most one predecessor and at most one successor of each type.

Now consider a triangle demand at u, with boundary $\{c_{S_p}, c_{S_u}, c_{S_w}\}$. Only c_{S_p} can be shared with a predecessor. If $p \in B_2$, this predecessor must be the demand at p. If p has one DFS child, it must be a triangle demand at $\mathrm{par}(p)$ whose lower vertex is p. These alternatives are exclusive, so there is at most one predecessor. Each of c_{S_u} and c_{S_w} can be shared with at most one successor, and Cases 2 and 3 show that both successors must be triangle demands.

Finally, suppose that K contains a cycle. A vertex of greatest DFS depth on that cycle would have two predecessors, contradicting the bound just proved. Hence K is a forest. Every vertex of $K[\mathcal{U}]$ has at most one predecessor and one successor, so its components are paths. ■

The procedure in Section G.3 uses at most $\rho = |V(K)| - \nu(K)$ paid increases. Since K is a forest, Kőnig's theorem gives

$$\rho = \alpha(K),$$

where $\alpha(K)$ is its independence number. Figure 24 illustrates the matching used by the procedure.

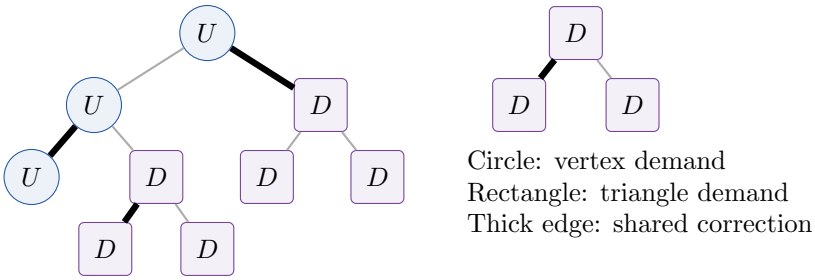


FIGURE 24. A correction forest with a maximum matching shown by thick edges. Each matched pair uses one correction; each unmatched demand uses its own connector. Circles denote vertex demands and rectangles triangle demands.

G.7. **Counting the corrections.** The correction procedure uses at most

$$\rho = |V(K)| - \nu(K)$$

paid increases. We want to prove that $\rho \le (n-2)/5$.

Since K is a forest, König's theorem implies that ρ is the maximum size of an independent set of K. It therefore suffices to prove $5|I| \le n-2$ for every independent set I.

Let $c_{\mathcal{U}}$ be the number of connected components of $K[\mathcal{U}]$, each of which is a path, possibly consisting of a single vertex. Let $c_{\mathcal{D}}$ be the number of connected components of K whose root belongs to $\mathcal{D}$. We prove

$$5|I| \le 3|\mathcal{U}| + 4|\mathcal{D}| + 2c_{\mathcal{U}} + c_{\mathcal{D}} \le n-2.$$

The first inequality uses the structure of K. The second follows by counting the edges and vertices of the DFS tree required by these demands.

Bounding the size of an independent set. Fix an independent set I. On each path of $K[\mathcal{U}]$, at most one more vertex belongs to I than to its complement. Thus

$$|I \cap \mathcal{U}| \le |\mathcal{U} \setminus I| + c_{\mathcal{U}},$$

or equivalently,

$$2|I \cap \mathcal{U}| \le |\mathcal{U}| + c_{\mathcal{U}}.$$

Every vertex in $I \cap \mathcal{D}$ either is the root of its component or has a predecessor outside I. A predecessor in $\mathcal{U}$ has at most one successor in $\mathcal{D}$, and a predecessor in $\mathcal{D}$ has at most two. Therefore,

$$|I \cap \mathcal{D}| \le |\mathcal{U} \setminus I| + 2|\mathcal{D} \setminus I| + c_{\mathcal{D}}.$$

Using $|\mathcal{U} \setminus I| = |\mathcal{U}| - |I \cap \mathcal{U}|$ and $|\mathcal{D} \setminus I| = |\mathcal{D}| - |I \cap \mathcal{D}|$, we obtain

$$|I \cap \mathcal{D}| \le |\mathcal{U}| - |I \cap \mathcal{U}| + 2|\mathcal{D}| - 2|I \cap \mathcal{D}| + c_{\mathcal{D}}.$$

Rearranging gives

$$|I \cap \mathcal{U}| + 3|I \cap \mathcal{D}| \le |\mathcal{U}| + 2|\mathcal{D}| + c_{\mathcal{D}}.$$

Combining these two inequalities with $|I \cap \mathcal{D}| \le |\mathcal{D}|$, we obtain

$$\begin{aligned} 5|I| &= 2\big(2|I \cap \mathcal{U}|\big) + \big(|I \cap \mathcal{U}| + 3|I \cap \mathcal{D}|\big) + 2|I \cap \mathcal{D}| \\ &\le 2\big(|\mathcal{U}| + c_{\mathcal{U}}\big) + |\mathcal{U}| + 2|\mathcal{D}| + c_{\mathcal{D}} + 2|\mathcal{D}| \\ &= 3|\mathcal{U}| + 4|\mathcal{D}| + 2c_{\mathcal{U}} + c_{\mathcal{D}}. \end{aligned}$$

Relating the demands to the size of G. We now prove the second inequality,

$$3|\mathcal{U}| + 4|\mathcal{D}| + 2c_{\mathcal{U}} + c_{\mathcal{D}} \le n-2.$$

For $i \in \{1, 2\}$, let n_i count the nonroot vertices with exactly one DFS child and exactly i back edges ending at them. Of the $n + 1$ back edges, three end at the root. Counting the remaining upper endpoints gives

$$n - 2 = n_1 + 2n_2 + |B_2|. \tag{48}$$

There are $1 + n_1 + n_2 + |B_2|$ stars, including the root star. Each contains one tree edge, and every other tree edge belongs to N. Hence

$$n - 1 = |N| + 1 + n_1 + n_2 + |B_2|.$$

Comparing this with (48) gives

$$n_2 = |N|, \qquad n - 2 = 2|N| + n_1 + |B_2|. \tag{49}$$

We bound $|N|$ and $n_1 + |B_2|$ separately.

The nonremovable edges. We claim that

$$|N| \geq |\mathcal{U}| + |\mathcal{D}| + c_{\mathcal{U}}. \tag{50}$$

Every demand center belongs to B_2 and has two DFS children. The edge to one child belongs to its star; the edge to the other is nonremovable. These give $|\mathcal{U}| + |\mathcal{D}|$ distinct edges of N.

For each path of $K[\mathcal{U}]$, let u be its highest vertex and put $p := \mathrm{par}(u)$. Since $u \in \mathcal{U}$, the edge pu is nonremovable. We show that p is not a demand center, so this edge has not already been counted.

If $p \in \mathcal{U}$, then pu gives an edge of $K[\mathcal{U}]$, contradicting the choice of u. If $p \in \mathcal{D}$, then $pu \in N$ makes u the lower vertex of the triangle demand at p. That vertex has exactly one DFS child, whereas $u \in \mathcal{U}$ has two. Both possibilities are therefore excluded.

Different path roots have different parent edges. These give the additional $c_{\mathcal{U}}$ edges in (50).

Counting vertices. We show that

$$n_1 + |B_2| \geq |\mathcal{U}| + 2|\mathcal{D}| + c_{\mathcal{D}}. \tag{51}$$

For each $u \in \mathcal{D}$, let w_u be the lower vertex of its triangle demand. Define

$$\begin{aligned} A_1 &:= \mathcal{U} \cup \mathcal{D}, \\ A_2 &:= \{w_u : u \in \mathcal{D}\}, \\ A_3 &:= \{\mathrm{par}(u) : u \in \mathcal{D},\ u \text{ is a component root in } K\}. \end{aligned}$$

All vertices of A_1 belong to B_2, and $|A_1| = |\mathcal{U}| + |\mathcal{D}|$. Every vertex of A_2 has exactly one DFS child and one back edge ending there, so $A_1 \cap A_2 = \varnothing$. Since $\mathrm{par}(w_u) = u$, different triangle demands have different lower vertices. Thus $|A_2| = |\mathcal{D}|$.

Consider $p = \mathrm{par}(u) \in A_3$, where the demand at u is a component root in K. The star $S_p = \{pu, pw_u\}$ has size two, so either $p \in B_2$ or p has exactly one DFS child and one back edge ending there. If $p \in A_1$, the demands at p and u share c_{S_p}, making p a predecessor of u in K. If $p \in A_2$, write $p = w_a$ for some $a \in \mathcal{D}$. Then $a = \mathrm{par}(p)$, and the demands at a and u share c_{S_p}, making a a predecessor of u. Both contradict the root condition. Hence $A_3 \cap (A_1 \cup A_2) = \varnothing$.

Different root triangle demands also have different parents: the unique star at a parent contains exactly one tree edge, which determines the center of the triangle below it. Therefore $|A_3| = c_{\mathcal{D}}$.

The three sets are pairwise disjoint. Every vertex of $(A_1 \cup A_2 \cup A_3) \setminus B_2$ has exactly one DFS child and one back edge ending there, so

$$|(A_1 \cup A_2 \cup A_3) \setminus B_2| \leq n_1.$$

Consequently,

$$n_1 + |B_2| \geq |A_1| + |A_2| + |A_3| = |\mathcal{U}| + 2|\mathcal{D}| + c_{\mathcal{D}},$$

as required.

Using the two counts in (49), we obtain

$$\begin{aligned} n-2 &= 2|N| + n_1 + |B_2| \\ &\geq 2\big(|\mathcal{U}| + |\mathcal{D}| + c_{\mathcal{U}}\big) + |\mathcal{U}| + 2|\mathcal{D}| + c_{\mathcal{D}} \\ &= 3|\mathcal{U}| + 4|\mathcal{D}| + 2c_{\mathcal{U}} + c_{\mathcal{D}}. \end{aligned} \tag{52}$$

Together with the independent-set bound, this gives $5|I| \leq n-2$. Taking an independent set of maximum size proves

$$\rho \leq \frac{n-2}{5}. \tag{53}$$

G.8. **The resulting tour bound.** *Proof.* The final vector $\bar{x}$ belongs to $\mathcal{P}^\circ$ by Proposition G.6. The objective expression at x^{init} is $5n/4 - 1/2$. The changes to edges in R_H leave it unchanged, and each paid increase raises it by $1/4$. Since at most $(n-2)/5$ paid increases are needed,

$$\begin{aligned} \operatorname{tsp}(G) &\leq \textsc{StarReconnectionLP}_{\text{OPT}}(G, \mathcal{S}) \leq \Phi_{G,\mathcal{S}}(\bar{x}) \\ &\leq \frac{5}{4}n - \frac{1}{2} + \frac{1}{4}\rho \\ &\leq \frac{5}{4}n - \frac{1}{2} + \frac{n-2}{20} \\ &= \frac{13}{10}n - \frac{3}{5}. \end{aligned}$$

The demands, the graph K, and a maximum matching in K can all be found in polynomial time. The StarReconnectionLP tour recovery then produces a tour satisfying this bound. ■

Appendix H. A 4/3 approximation for k-connected graphs with $\Delta \le k+1$

We now construct a point in the feasible region of StarDeletionLP for simple k-vertex-connected graphs of maximum degree at most $k+1$, where $k \ge 3$. This includes k-connected, k-regular graphs.

Theorem H.1. *Let $k \ge 3$ and let $G = (V, E)$ be a simple k-vertex-connected graph with $\Delta(G) \le k+1$. Fix any DFS tree T, and let $\mathcal{S}$ be the* Star *removable system constructed from T as in Section D. Then*

$$\begin{aligned} \operatorname{tsp}(G) &\le \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}) \\ &\le \textsc{StarDeletionLP}_{\mathrm{OPT}}(G, \mathcal{S}) \le \frac{4}{3}n - \frac{2}{3}. \end{aligned} \tag{54}$$

A tour satisfying this bound can be computed in polynomial time.

Proof. We construct $y \in \mathcal{P}^{\mathrm{SDL}}(G, \mathcal{S})$ and prove that $\Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(y) = 4n/3 - 2/3$.

The edge values. Let r be the DFS root. For a star $S = S(u, v)$, write $t_S := |I(u, v)|$, so $k_S = t_S + 1$. If $u \ne r$, the parent edge of u and its child edge uv are distinct from these back edges. Hence

$$1 \le t_S \le \deg_G(u) - 2 \le k - 1.$$

Since G is 2-vertex-connected, the root has exactly one DFS child. Its star S_r therefore contains all edges incident to r, and

$$t_{S_r} = \deg_G(r) - 1 \in \{k-1, k\}.$$

In particular, $t_{S_r} \ge 2$.

For $e \in E(G)$ and $S \in \mathcal{S}$, define

$$y_{\mathrm{rep}(e)} := \begin{cases} \dfrac{1}{3}, & e \in E(T), \\ 1 - \dfrac{2}{3t_S}, & e \in I(u, v), \quad S = S(u, v) \ne S_r, \\ 1 - \dfrac{4}{3t_{S_r}}, & e \in S_r \setminus E(T), \end{cases} \tag{55}$$

$$y_{c_S} := \begin{cases} \dfrac{1}{3}, & S \ne S_r, \\ 1, & S = S_r. \end{cases}$$

Every coordinate lies in $[1/3, 1]$.

The auxiliary degree bounds. The different values at the root ensure that every star satisfies its degree bound with equality:

$$y(S_{H^{\mathrm{star}}}) + y_{c_S} = \frac{1}{3} + t_S\left(1 - \frac{2}{3t_S}\right) + \frac{1}{3} = t_S = k_S - 1 \qquad (S \ne S_r),$$

$$y((S_r)_{H^{\mathrm{star}}}) + y_{c_{S_r}} = \frac{1}{3} + t_{S_r}\left(1 - \frac{4}{3t_{S_r}}\right) + 1 = t_{S_r} = k_{S_r} - 1.$$

The Blossom inequalities. We first record a bound used in the cut calculations. For a back edge b in a nonroot star S,

$$1 - y_{\mathrm{rep}(b)} = \frac{2}{3t_S} \ge \frac{2}{3(k-1)}.$$

For a back edge in the root star,

$$1 - y_{\mathrm{rep}(b)} = \frac{4}{3t_{S_r}} \ge \frac{4}{3k} \ge \frac{2}{3(k-1)}.$$

Thus every back edge satisfies

$$1 - y_{\mathrm{rep}(b)} \geq \frac{2}{3(k-1)}. \tag{56}$$

Consider a cut $C := \delta_{H^{\mathrm{star}}}(U)$ and $F \subseteq C$ with $|C \setminus F|$ odd. If $|C \setminus F| \geq 3$, then $y(C \setminus F) \geq 1$, which proves its Blossom inequality. Otherwise, $C \setminus F = \{f\}$, and the required inequality is equivalent to

$$\sum_{a \in C} (1 - y_a) \geq 2(1 - y_f).$$

Since $y_f \geq 1/3$, it suffices to show that the sum is at least $4/3$.

The root connector has value one, so the inequality is immediate if $f = c_{S_r}$. Assume otherwise. The original root r is incident only to c_{S_r} in H^{star}. If this connector crosses the cut, move r to the shore containing z_{S_r}. This changes no other crossing edge and leaves the displayed sum unchanged. We may therefore contract c_{S_r}. The resulting cut is still nonempty, since f still crosses it.

The edges $\mathrm{rep}(e)$ with $e \in E(T)$ and the remaining connectors form a spanning tree, and each has value $1/3$. We call this the expanded tree and consider the number of its edges crossing the cut.

Case 1: At least two expanded-tree edges cross. These two edges alone give

$$\sum_{a \in C} (1 - y_a) \geq \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

Case 2: Exactly one expanded-tree edge crosses. The two shores are the vertex sets of the two components obtained by deleting this edge from the expanded tree. We show that at least $k - 1$ back edges also cross.

If the crossing edge represents a tree edge uv, consider the original subtree T_v. The other crossing edges correspond exactly to the back edges in $\delta_G(V(T_v))$. Since G is k-edge-connected, this original cut has at least k edges. Only uv is a tree edge, so at least $k - 1$ back edges cross.

If the crossing edge is a connector $c_S = uz_S$, where $S = S(u, v)$ and $u \neq r$, the component below it contains z_S and the subtree corresponding to T_v in H^{star}. For every $b \in I(u, v)$, both endpoints of $\mathrm{rep}(b)$ lie in this component. Thus, apart from c_S, the crossing edges correspond exactly to the back edges of G from $V(T_v)$ to proper ancestors of u.

To count these edges, delete u from G. They form the cut between $V(T_v)$ and $V \setminus (V(T_v) \cup \{u\})$ in $G - u$. Both sides are nonempty: one contains v, and the other contains r. The graph $G - u$ is $(k-1)$-vertex-connected and hence $(k-1)$-edge-connected. Thus at least $k - 1$ back edges cross in this case as well.

In either case, the single crossing expanded-tree edge and (56) give

$$\begin{aligned} \sum_{a \in C} (1 - y_a) &\geq \frac{2}{3} + (k-1)\frac{2}{3(k-1)} \\ &= \frac{4}{3}. \end{aligned}$$

These cases prove every Blossom inequality. Together with the coordinate and degree bounds, they show that $y \in \mathcal{P}^{\mathrm{SDL}}(G, \mathcal{S})$.

The objective. Every back edge belongs to exactly one star, and every star contains exactly one tree edge. Consequently,

$$\sum_{S \in \mathcal{S}} t_S = m - n + 1, \qquad |N| + |\mathcal{S}| = n - 1.$$

Using the degree equalities $y(S_{H^{\mathrm{star}}}) = t_S - y_{c_S}$ and $y(N_{H^{\mathrm{star}}}) = |N|/3$, we obtain

$$\begin{aligned}\Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(y) &= m + \frac{|N|}{3} - \sum_{S\in\mathcal{S}} (t_S - y_{c_S}) \\ &= n - 1 + \frac{|N|}{3} + \sum_{S\in\mathcal{S}} y_{c_S} \\ &= n - 1 + \frac{|N|}{3} + 1 + \frac{|\mathcal{S}| - 1}{3} \\ &= \frac{4}{3}n - \frac{2}{3}.\end{aligned}$$

Since $\textsc{StarDeletionLP}_{\mathrm{OPT}}(G,\mathcal{S}) \le \Phi^{\mathrm{SDL}}_{G,\mathcal{S}}(y)$, Theorem B.1 and Proposition C.3 give (54). The point is computable in polynomial time, as are $\textsc{StarDeletionLP}$ optimization and the corresponding tour recovery. ■

Appendix I. Analysis for arbitrary subtour value

Let (G_0, x^*) be the input to StarMatch, where x^* is an optimal extreme subtour solution whose positive support is 2-vertex-connected. Write $G = (V, E)$ for this support, $n := |V|$, and $m := |E|$. If G is a cycle, StarMatch returns it directly. Otherwise, let x be the box-constrained SubtourLP solution obtained from x^* by StarMatch on the same positive support, with

$$0 < x_e \le 1 \quad (e \in E), \qquad x(E) = x^*(E) = \textsc{SubtourLP}(G).$$

We use this vector x throughout the section. The extremality of x^* gives $m \le 2n - 3$, although x need not be extreme.

Theorem I.1. *Let T be the greedy DFS tree constructed by StarMatch on G, and let $\mathcal{S}$ be the Star removable system constructed from T as in Section D. Then*

$$\begin{aligned} \mathrm{tsp}(G) &\le \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G, \mathcal{S}) \\ &\le \frac{89}{64} n + \frac{9}{8}(\textsc{SubtourLP}(G) - n) - \frac{13}{16} \\ &= \frac{89}{64}\textsc{SubtourLP}(G) - \frac{17}{64}(\textsc{SubtourLP}(G) - n) - \frac{13}{16}. \end{aligned} \tag{57}$$

Together with the standard block reduction, this gives a polynomial-time 89/64-approximation relative to the subtour optimum for every connected simple graph.

The algorithm roots the greedy DFS at a vertex r with $\deg_G(r) \ge 3$. For the calculations, write $L := \textsc{SubtourLP}(G)$ and put

$$\Delta := L - n,$$
$$\eta_v := x(\delta_G(v)) - 2 \ge 0, \quad \sum_{v \in V} \eta_v = 2\Delta.$$

For a star $S = S(u, v)$, put

$$q_S := x_{uv}, \qquad a_S := x(I(u, v)), \qquad t_S := |I(u, v)|, \qquad k_S := t_S + 1, \qquad e_S := (a_S - 1)_+,$$

where $(z)_+ := \max\{z, 0\}$. Thus q_S is the value of the unique tree edge of S, while a_S and t_S are the total x-value and number of its back edges. Call S positive when $e_S > 0$. Greedy DFS gives $a_S \le t_S q_S$. Let $o(v)$ be the total x-value of back edges starting at v and ending at its proper ancestors. For consecutive stars $S = S(u, v)$ and $S' = S(v, w)$, the value of the parent edge of the center v is $q_S = x_{uv}$. Figure 25 illustrates the quantities associated with S.

Every back edge belongs to exactly one star, and each star contains one tree edge. Hence

$$|N| + |\mathcal{S}| = n - 1, \qquad m - \sum_{S \in \mathcal{S}} (k_S - 1) = n - 1.$$

By (27), the objective is therefore

$$\Phi_{G,\mathcal{S}}(y) = n - 1 + y(N_H) + \sum_{S \in \mathcal{S}} \max\{y_{c_S},\, k_S - 1 - y(S_H)\}. \tag{58}$$

Corollary C.2 bounds the tour returned by StarMatch by this value for every $y \in \mathcal{P}^\circ$.

The root has one DFS child and at least two back edges. Every back edge belongs to exactly one star, so

$$\sum_{S \ne S_r} t_S = m - n + 1 - t_{S_r} \le n - 4. \tag{59}$$

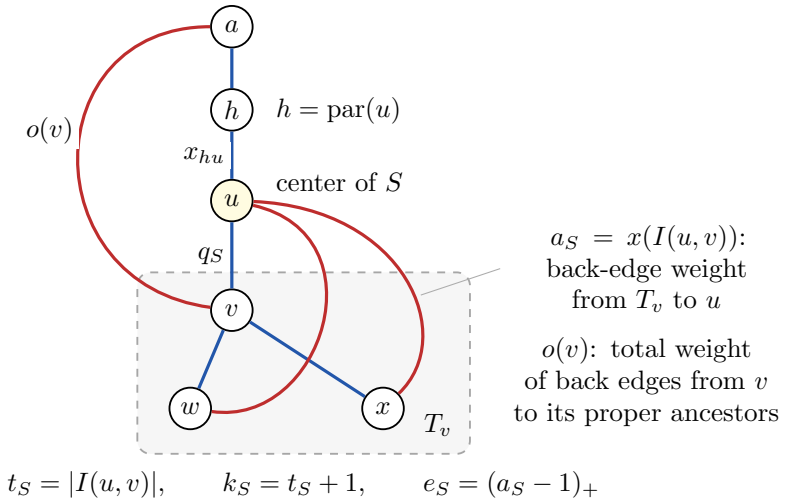


FIGURE 25. The star $S = S(u, v)$ in G, with the subtree T_v shaded. Blue edges are tree edges; red edges are back edges. The tree edge uv has value $q_S = x_{uv}$, and the back edges in $I(u, v)$ have total x-value a_S.

I.1. **The point $Q(d)$ and its connector values.** For every nonroot star S, define

$$\bar{e}_S := \min\{e_S, 1 - q_S\}.$$

Define $Q : \mathbb{R}_+^{\mathcal{S}} \to \mathbb{R}_+^{E(H^\circ)}$ by specifying its values on the nonconnector edges and then on the connectors. We call $d \in \mathbb{R}_+^{\mathcal{S}}$ admissible when

$$d_{S_r} = 0, \qquad 0 \le d_S \le \bar{e}_S \quad (S \ne S_r).$$

For any $d \in \mathbb{R}_+^{\mathcal{S}}$, set

$$Q(d)_{\mathrm{rep}(e)} := \begin{cases} \dfrac{1}{3} + \dfrac{2}{3} d_S, & e = uv, \quad S = S(u, v) \ne S_r, \\ 1 - \dfrac{2}{3} x_e, & e \notin E(T), \\ \dfrac{1}{3}, & \text{otherwise.} \end{cases}$$

The connector assignment in (10) is specific to $\textsc{SubtourLP}(G) = n$. For arbitrary L, we use minimum cuts to choose the connector values and prove the Blossom inequalities.

Set $Q(d)_{c_{S_r}} := 1$ and let K be the graph obtained by contracting the root connector in H°. Using the values $Q(d)_{\mathrm{rep}(e)}$ already defined above, assign capacity $1 - Q(d)_{\mathrm{rep}(e)}$ to each edge $\mathrm{rep}(e)$ of K and capacity $2/3$ to each remaining connector. These capacities are nonnegative when d is admissible.

For each nonroot star S centered at v, define $\kappa_S(d)$ by the following finite minimum over cuts separating v from z_S in $K - c_S$, where $c_S = vz_S$:

$$\kappa_S(d) := \min_{\substack{U \subseteq V(K): \\ v \in U,\ z_S \notin U}} \left\{ \sum_{\substack{e \in E(G): \\ \mathrm{rep}(e) \in \delta_{K - c_S}(U)}} \left(1 - Q(d)_{\mathrm{rep}(e)}\right) + \frac{2}{3} \left| \{S' \in \mathcal{S} \setminus \{S_r\} : c_{S'} \in \delta_{K - c_S}(U)\} \right| \right\}. \tag{60}$$

This minimum is well-defined for every $d \in \mathbb{R}_+^{\mathcal{S}}$; for admissible d, it is a minimum-cut value with nonnegative capacities. Complete the definition of $Q(d)$ by setting

$$Q(d)_{c_S} := \begin{cases} 1, & S = S_r, \\ \max\left\{\frac{1}{3},\, 1 - \kappa_S(d)\right\}, & S \neq S_r. \end{cases}$$

A nonroot connector therefore remains at $1/3$ when $\kappa_S(d) \geq 2/3$. Otherwise, we increase its value so that

$$1 - Q(d)_{c_S} = \kappa_S(d).$$

For arbitrary nonnegative d, these formulas define a nonnegative vector. The next lemma proves feasibility for admissible d.

Lemma I.2. *For every admissible $d \in \mathbb{R}_+^{\mathcal{S}}$, the vector $Q(d)$ belongs to $\mathcal{P}^\circ$.*

Proof. For admissible d and $S = S(u,v) \neq S_r$, the restriction $d_S \leq 1 - q_S$ gives

$$1 - Q(d)_{\mathrm{rep}(uv)} = \frac{2}{3}(1 - d_S) \geq \frac{2}{3} q_S = \frac{2}{3} x_{uv}.$$

The same inequality holds with equality for back edges, and follows from $x_e \leq 1$ for the remaining edges. Thus

$$1 - Q(d)_{\mathrm{rep}(e)} \geq \frac{2}{3} x_e \qquad (e \in E(G)).$$

We use this inequality to bound the terms in the Blossom inequalities for $Q(d)$ using the subtour cut constraints for x.

Each $\kappa_S(d)$ is the value of a minimum cut with nonnegative capacities. All coordinates of $Q(d)$ lie in $[1/3, 1]$. By (26), it remains to prove the Blossom inequalities.

Consider $C := \delta_{H^\circ}(U)$ and $F \subseteq C$ with $|C \setminus F|$ odd. If $|C \setminus F| \geq 3$, then

$$Q(d)(C \setminus F) + |F| - Q(d)(F) \geq \frac{|C \setminus F|}{3} \geq 1.$$

We therefore need only consider $C \setminus F = \{f\}$, when the Blossom inequality is equivalent to

$$1 - Q(d)_f \leq \sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big). \tag{61}$$

If $f = c_{S_r}$, its left-hand side is zero. Otherwise, place the degree-one vertex r on the same side of the cut as z_{S_r}. This changes no crossing edge except possibly c_{S_r}, whose complement is zero. It therefore suffices to prove (61) for cuts of K.

Suppose the inequality fails for some edge f. Among the cuts violating it for this same f, choose $C = \delta_K(U)$ with the fewest crossing connectors. We consider three cases.

Case 1: No connector crosses C. Contracting all connectors identifies C with the edges of H° corresponding to a nontrivial cut $\delta_G(U_G)$ of G. The subtour constraint therefore gives

$$\begin{aligned} \sum_{a \in C} \big(1 - Q(d)_a\big) &= \sum_{e \in \delta_G(U_G)} \big(1 - Q(d)_{\mathrm{rep}(e)}\big) \\ &\geq \frac{2}{3} x(\delta_G(U_G)) \geq \frac{4}{3}. \end{aligned}$$

Since $1 - Q(d)_f \leq 2/3$, we obtain

$$\begin{aligned} \sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big) &\geq \frac{4}{3} - \big(1 - Q(d)_f\big) \\ &\geq 1 - Q(d)_f. \end{aligned}$$

This contradicts the choice of C as a violating cut.

Case 2: A connector crosses C, and every connector in $C \setminus \{f\}$ has value $1/3$. Every connector $c_{S'} \in C \setminus \{f\}$ satisfies

$$1 - Q(d)_{c_{S'}} = \frac{2}{3},$$

which equals its capacity in the minimum-cut calculation. For edges of the form $\mathrm{rep}(e)$, the capacity is $1 - Q(d)_{\mathrm{rep}(e)}$ by definition. Thus every edge in $C \setminus \{f\}$ has capacity equal to its complement.

Suppose first that $f = \mathrm{rep}(e)$ for some $e \in E(G)$. Choose a connector $c_{S'}$ crossing C. Since $c_{S'} \neq f$, the sum over $C \setminus \{f\}$ contains the term $1 - Q(d)_{c_{S'}} = 2/3$. Also, $Q(d)_f \geq 1/3$, so

$$\sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big) \geq 1 - Q(d)_{c_{S'}} = \frac{2}{3} \geq 1 - Q(d)_f.$$

Now suppose that $f = c_S = vz_S$. Since $C = \delta_K(U)$ contains c_S, the cut

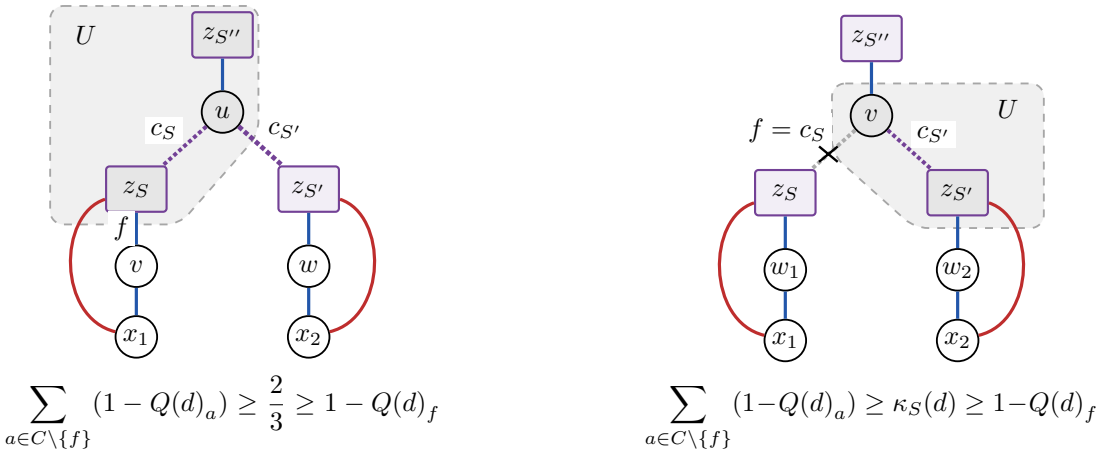
$$C \setminus \{f\} = \delta_{K - c_S}(U)$$

separates v from z_S in $K - c_S$. Every connector $c_{S'} \in C \setminus \{f\}$ has value $1/3$, so its complement $1 - Q(d)_{c_{S'}} = 2/3$ equals its assigned capacity. The capacity of each representative edge also equals its complement by definition.

Recall that $\kappa_S(d)$ is the minimum capacity of a cut separating v from z_S in $K - c_S$, as defined in (60). Therefore,

$$\begin{aligned} \sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big) &\geq \kappa_S(d) \\ &\geq \min\left\{\frac{2}{3}, \kappa_S(d)\right\} \\ &= 1 - Q(d)_f, \end{aligned}$$

where the equality follows from the definition of $Q(d)_{c_S}$. Thus (61) holds in both situations, contradicting the choice of C.



(A) $f = \mathrm{rep}(uv)$ and $Q(d)_{c_{S'}} = 1/3$.

(B) $C \setminus \{f\} = \delta_{K - c_S}(U)$.

FIGURE 26. The two situations in Case 2. Shading marks U. In (b), c_S is deleted for the minimum-cut calculation. Blue edges and dotted connectors follow the DFS tree; red curves correspond to back edges.

Case 3: A connector in $C \setminus \{f\}$ has value greater than $1/3$. Choose such a connector $e = c_S$. Let W attain the minimum in (60), and put $D := \delta_K(W)$. Then $e \in D$, and $D \setminus \{e\}$ has capacity $\kappa_S(d)$. Since $Q(d)_e > 1/3$, the definition of its coordinate gives

$$\kappa_S(d) = 1 - Q(d)_e < \frac{2}{3}.$$

Every connector is assigned capacity $2/3$ in the minimum-cut construction. Since the total capacity of $D \setminus \{e\}$ is less than $2/3$, this set contains only representative edges. Their capacities equal their complements, so

$$\sum_{a \in D \setminus \{e\}} \big(1 - Q(d)_a\big) = \kappa_S(d) = 1 - Q(d)_e.$$

We use this equality to show that $f \notin D$. Since $e \in C \setminus \{f\}$ and C violates (61),

$$1 - Q(d)_f > \sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big) \geq 1 - Q(d)_e.$$

On the other hand, the equality above gives

$$1 - Q(d)_a \leq 1 - Q(d)_e \qquad (a \in D \setminus \{e\}).$$

Thus $f \notin D \setminus \{e\}$, and since $f \neq e$, we obtain $f \notin D$.

Now replace C by the cut

$$C' := C \triangle D = \delta_K(U \triangle W).$$

The edge f remains in the cut, while e leaves it. Moreover,

$$\begin{aligned}
\sum_{a \in C' \setminus \{f\}} \big(1 - Q(d)_a\big) &\leq \sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big) - \big(1 - Q(d)_e\big) \\
&\quad + \sum_{a \in D \setminus \{e\}} \big(1 - Q(d)_a\big) \\
&= \sum_{a \in C \setminus \{f\}} \big(1 - Q(d)_a\big) \\
&< 1 - Q(d)_f.
\end{aligned}$$

Thus C' still violates (61) for f. But D contains no connector other than e, so C' has exactly one fewer crossing connector than C. This contradicts the choice of C.



(A) $C = \delta_K(U)$. (B) $C' = \delta_K(U \triangle W)$.

FIGURE 27. Case 3. The shaded sets are U and $U \triangle W$, respectively; the dashed outline marks W. The connector e stops crossing, while f remains in the cut.

The three cases cover every possibility for C, and each contradicts its choice as a violating cut. Thus (61) holds for all cuts of K and, by the root-connector reduction above, for all cuts of H°. All Blossom inequalities hold, so $Q(d) \in \mathcal{P}^\circ$. ■

I.2. **How much can a connector increase?** For a nonroot star S centered at v, define

$$\tau_S := \big(1 - \max\{0, 2 - q_S - a_S, q_S + a_S - \eta_v\}\big)_+. \tag{62}$$

We use the subtour constraints to bound the capacities of the cuts in (60). Together with the cuts that isolate v or z_S after contracting the other connectors, these give the following bound.

Lemma I.3. *Let $d \in \mathbb{R}^{\mathcal{S}}_+$ be admissible. Let S be a nonroot star centered at v, and let $u := \mathrm{par}(v)$. Then*

$$Q(d)_{c_S} \le \begin{cases} \dfrac{1}{3} + \dfrac{2}{3}\max\{\tau_S, (d_{S^{\mathrm{par}}} - o(v))_+\}, & \text{if } v \text{ has exactly one DFS child and} \\ & S^{\mathrm{par}} = S(u,v) \in \mathcal{S} \setminus \{S_r\}, \\ \dfrac{1}{3} + \dfrac{2}{3}\tau_S, & \text{if } v \text{ has at least two DFS children} \\ & \text{or } uv \in N \cup S_r. \end{cases} \tag{63}$$

Proof. By the definition of the connector coordinates, $1 - Q(d)_{c_S} = \min\{2/3, \kappa_S(d)\}$. We therefore prove the lemma by bounding the sums in the minimum defining $\kappa_S(d)$ in (60).

Let $C := \delta_{K - c_S}(U)$ be any cut with $v \in U$ and $z_S \notin U$. If C contains a connector, its term in that sum is $2/3$, which already gives the required bound. We may therefore suppose that C contains no connector. The sum to be bounded is then $\sum_{a \in C}(1 - Q(d)_a)$.

If v has at least two DFS children or $uv \in N \cup S_r$, we prove

$$\sum_{a \in C}\big(1 - Q(d)_a\big) \ge \frac{2}{3}(1 - \tau_S).$$

If v has exactly one DFS child and its parent edge belongs to a nonroot star $S^{\mathrm{par}} = S(u,v)$, it suffices to prove

$$\sum_{a \in C}\big(1 - Q(d)_a\big) \ge \frac{2}{3}\left(1 - \max\{\tau_S, (d_{S^{\mathrm{par}}} - o(v))_+\}\right).$$

Taking the minimum over the cuts gives the required bound on $\kappa_S(d)$ and hence on $Q(d)_{c_S}$.

Contract all remaining connectors in $K - c_S$. None crosses C, so the sum above does not change. The resulting graph is obtained from G by adding z_S and replacing each edge $vw \in S$ by $z_S w$. Write the two sides as $A \cup \{v\}$ and $B \cup \{z_S\}$, where A, B partition $V(G) \setminus \{v\}$. We consider three cases.

Case 1: $A \neq \varnothing$ and $B \neq \varnothing$. The subtour constraints for B and A, together with $x(\delta_G(v)) = 2 + \eta_v$, give

$$x(E_G(A,B)) + x(E_G(\{v\},B)) \geq 2,$$
$$x(E_G(A,B)) + 2 + \eta_v - x(E_G(\{v\},B)) \geq 2.$$

After the contractions, the edges of S_H are incident to z_S, while the edges $\mathrm{rep}(e)$ with $e \in \delta_G(v) \setminus S$ are incident to v. Thus an edge of S_H crosses C precisely when its other endpoint lies in A, and an edge incident to v crosses precisely when its other endpoint lies in B. The edges corresponding to $E_G(A,B)$ also cross C. Using $1 - Q(d)_{\mathrm{rep}(e)} \geq 2x_e/3$, we obtain

$$\sum_{a \in C} \big(1 - Q(d)_a\big) \geq \frac{2}{3}\Big(x(E_G(A,B)) + x(S \cap E_G(\{v\},A)) + x(E_G(\{v\},B) \setminus S)\Big).$$

To prove the lemma in this case, it suffices to show that

$$\begin{aligned} 1 - \tau_S &\leq \max\{0, 2 - q_S - a_S, q_S + a_S - \eta_v\} \\ &\leq x(E_G(A,B)) + x(S \cap E_G(\{v\},A)) + x(E_G(\{v\},B) \setminus S). \end{aligned}$$

We bound each term in the maximum separately.

The term 0. Every term on the right is nonnegative.

The term $2 - q_S - a_S$. The cut $\delta_G(B)$ consists of the edges between A and B and the edges between v and B. Therefore,

$$\begin{aligned} &x(E_G(A,B)) + x(S \cap E_G(\{v\},A)) + x(E_G(\{v\},B) \setminus S) \\ &\quad \geq x(E_G(A,B)) + x(E_G(\{v\},B) \setminus S) \\ &\quad = x(\delta_G(B)) - x(S \cap E_G(\{v\},B)) \\ &\quad \geq 2 - x(S) \\ &\quad = 2 - q_S - a_S. \end{aligned}$$

Here we used the subtour constraint for B and $S \cap E_G(\{v\},B) \subseteq S$.

The term $q_S + a_S - \eta_v$. We use the cut $\delta_G(A)$ in the same way. The edges incident to v outside S have total x-value

$$x(\delta_G(v) \setminus S) = x(\delta_G(v)) - x(S) = 2 + \eta_v - q_S - a_S.$$

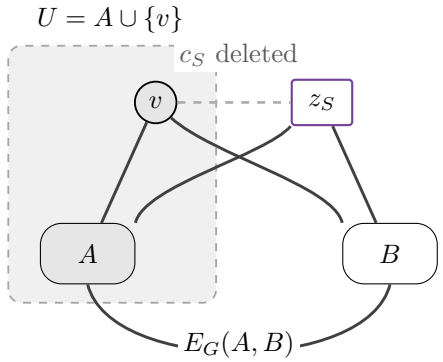
Consequently,

$$\begin{aligned} &x(E_G(A,B)) + x(S \cap E_G(\{v\},A)) + x(E_G(\{v\},B) \setminus S) \\ &\quad \geq x(E_G(A,B)) + x(S \cap E_G(\{v\},A)) \\ &\quad = x(\delta_G(A)) - x(E_G(\{v\},A) \setminus S) \\ &\quad \geq 2 - x(\delta_G(v) \setminus S) \\ &\quad = 2 - (2 + \eta_v - q_S - a_S) \\ &\quad = q_S + a_S - \eta_v. \end{aligned}$$

Here we used the subtour constraint for A and $E_G(\{v\},A) \setminus S \subseteq \delta_G(v) \setminus S$.

Combining these bounds with the preceding estimate on the capacity of C gives

$$\sum_{a\in C}\bigl(1-Q(d)_a\bigr) \ge \frac{2}{3}\max\{0, 2-q_S-a_S, q_S+a_S-\eta_v\}$$
$$\ge \frac{2}{3}(1-\tau_S).$$



The edges from z_S to A belong to S_H.

The edges from v to B correspond to edges outside S.

$$\sum_{a\in C}\bigl(1-Q(d)_a\bigr) \ge \frac{2}{3}(1-\tau_S).$$

FIGURE 28. Case 1 after contracting the other connectors and deleting c_S. Shading marks $A\cup\{v\}$; boxes denote sets of vertices and solid lines denote sets of edges. The subtour constraints for A and B give the capacity bound.

Case 2: The cut isolates z_S. Here $B = \varnothing$, and the crossing edges are exactly S_H. Their total capacity is

$$\sum_{a\in S_H}\bigl(1-Q(d)_a\bigr) = \frac{2}{3}(1-d_S+a_S)$$
$$\ge \frac{2}{3},$$

because $d_S \le \bar{e}_S \le e_S \le a_S$.

Case 3: The cut isolates v after contraction. Here $A = \varnothing$, and the crossing edges are

$$\{\mathrm{rep}(e) : e \in \delta_G(v)\setminus S\}.$$

Suppose v has exactly one DFS child and $S^{\mathrm{par}} = S(u,v) \in \mathcal{S}\setminus\{S_r\}$. The edges of $\delta_G(v)\setminus S$ are then uv and the back edges starting at v. By the definition of $Q(d)$,

$$\sum_{e\in\delta_G(v)\setminus S}\bigl(1-Q(d)_{\mathrm{rep}(e)}\bigr) = \bigl(1-Q(d)_{\mathrm{rep}(uv)}\bigr) + \sum_{\substack{b\notin E(T):\\ b \text{ starts at } v}}\bigl(1-Q(d)_{\mathrm{rep}(b)}\bigr)$$
$$= \frac{2}{3}(1-d_{S^{\mathrm{par}}}) + \frac{2}{3}o(v)$$
$$= \frac{2}{3}(1-d_{S^{\mathrm{par}}}+o(v)).$$

If $uv \in N\cup S_r$, then $\mathrm{rep}(uv)$ crosses the cut and $1-Q(d)_{\mathrm{rep}(uv)} = 2/3$. Thus the cut has capacity at least $2/3$.

Finally, suppose v has at least two DFS children, and choose a child w other than the child defining S. If $vw \in N$, then $\mathrm{rep}(vw)$ crosses the cut and has capacity $2/3$. If $S'' = S(v,w) \in \mathcal{S}$, its

connector has been contracted, so every edge of S''_H crosses the cut. Consequently,

$$\begin{aligned}\sum_{e\in\delta_G(v)\setminus S}\big(1-Q(d)_{\mathrm{rep}(e)}\big) &\geq \sum_{a\in S''_H}\big(1-Q(d)_a\big)\\ &= \frac{2}{3}(1-d_{S''}+a_{S''})\\ &\geq \frac{2}{3}.\end{aligned}$$

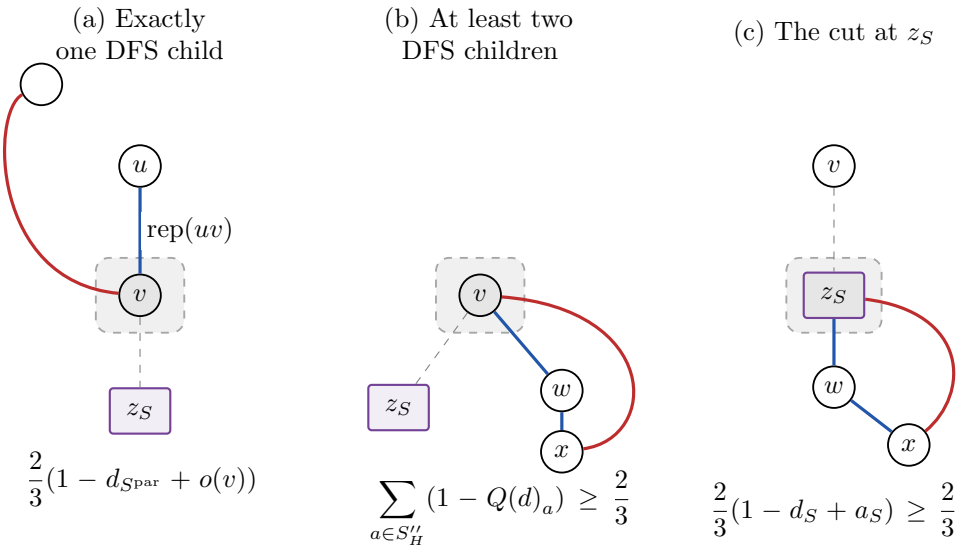


FIGURE 29. The singleton cuts after contracting the other connectors. Gray dashed edges mark the deleted connector c_S. The expressions below the panels give the cut capacity in (a) and (c), and a lower bound in (b). Curved edges may represent several back edges.

These cases cover every cut in the minimum defining $\kappa_S(d)$. Taking the minimum of their capacity bounds and using

$$Q(d)_{c_S} = 1-\min\left\{\frac{2}{3},\kappa_S(d)\right\}$$

gives (63). ∎

When $\eta_v = 0$, we have $\max\{2-q_S-a_S, q_S+a_S\}\geq 1$, so $\tau_S = 0$. Whenever the bound in (63) exceeds $1/3$, the singleton cut at v attains the minimum defining $\kappa_S(d)$, and hence the connector attains that bound. When the bound equals $1/3$, equality follows from $Q(d)_{c_S}\geq 1/3$. Thus (63) is an equality when $\eta_v = 0$.

I.3. The point $Q^{(2)}$.

Lemma I.4 (Lifting a join from G to H°)**.** *Let y belong to the* odd(G)*-join polytope. Then there exists $\widetilde{y}\in\mathcal{P}^\circ$ such that*

$$\widetilde{y}_{\mathrm{rep}(e)} = y_e \qquad (e\in E(G)).$$

Proof. By Theorem A.5, write

$$y=\sum_i \lambda_i \mathbf{1}_{J_i},\qquad \lambda_i\geq 0,\qquad \sum_i\lambda_i = 1,$$

where each J_i is an odd(G)-join. For each J_i, form an edge set $K_i \subseteq E(H^\circ)$ by selecting $\mathrm{rep}(e)$ exactly when $e \in J_i$ and choosing the connectors by

$$\mathbf{1}_{\{c_S \in K_i\}} \equiv k_S + 1 - |J_i \cap S| \pmod 2. \tag{64}$$

The parity argument in the proof of Lemma F.2 shows that K_i is an odd(H°)-join. Set

$$\widetilde{y} := \sum_i \lambda_i \mathbf{1}_{K_i}.$$

Convexity gives $\widetilde{y} \in \mathcal{P}^\circ$, and

$$\widetilde{y}_{\mathrm{rep}(e)} = \sum_i \lambda_i \mathbf{1}_{\{e \in J_i\}} = y_e.$$

Each integral join has a unique lift, whereas the fractional lift may depend on the chosen convex decomposition of y. ■

Define $Q_e^{(2),G} := x_e/2$ for $e \in E(T)$ and $Q_e^{(2),G} := 1 - x_e/2$ otherwise. Since $0 \le x_e \le 1$, the definition of $Q^{(2),G}$ gives

$$\min\{Q_e^{(2),G},\, 1 - Q_e^{(2),G}\} = \frac{x_e}{2} \ge 0 \qquad (e \in E(G)).$$

Consequently, for every nontrivial cut C of G and every $F \subseteq C$,

$$\sum_{e \in C \setminus F} Q_e^{(2),G} + \sum_{e \in F} (1 - Q_e^{(2),G}) \ge \frac{x(C)}{2} \ge 1.$$

Thus $Q^{(2),G}$ belongs to the odd(G)-join polytope. Lemma I.4 allows us to lift $Q^{(2),G}$ to a point of $\mathcal{P}^\circ$ whose value on each edge $\mathrm{rep}(e)$ is $Q_e^{(2),G}$ for $e \in E(G)$. This lift need not be unique; fix any such lift and denote it by $Q^{(2)}$. We next use local Blossom cut inequalities to bound its connector values. These bounds hold for every such lift. The cut at the DFS root r in H° consists only of c_{S_r}. Its Blossom inequality gives $Q^{(2)}_{c_{S_r}} \ge 1$, which implies $Q^{(2)}_{c_{S_r}} = 1$.

For a nonroot star $S = S(v, w)$, take $C := \delta_{H^\circ}(\{z_S\})$ and $F := \{c_S\} \cup \{\mathrm{rep}(b) : b \in I(v, w)\}$. Since $C \setminus F = \{\mathrm{rep}(vw)\}$, the Blossom inequality gives

$$1 \le \frac{q_S}{2} + (1 - Q^{(2)}_{c_S}) + \frac{a_S}{2}.$$

Hence

$$Q^{(2)}_{c_S} \le \frac{q_S + a_S}{2}. \tag{65}$$

When v has even degree in T, take

$$U := \{v\} \cup \{z_{S'} : S' \ne S \text{ is centered at } v\}, \qquad C := \delta_{H^\circ}(U).$$

The cut contains c_S and the edges $\mathrm{rep}(e)$ with $e \in \delta_G(v) \setminus S$. Choose

$$F := \{c_S\} \cup \{\mathrm{rep}(b) \in C : b \in E(G) \setminus E(T)\}.$$

The edges in $C \setminus F$ correspond to the edges of T incident to v, except for the one belonging to S. Consequently, $|C \setminus F| = \deg_T(v) - 1$ is odd, so we can apply the Blossom inequality in (26) for this choice of C and F.

Each edge $\mathrm{rep}(e) \in C \setminus \{c_S\}$ contributes $x_e/2$ to this inequality: through ${Q^{(2)}}_{\mathrm{rep}(e)}$ when $e \in E(T)$, and through $1 - {Q^{(2)}}_{\mathrm{rep}(e)}$ otherwise. Using $x(\delta_G(v)) = 2 + \eta_v$, we obtain

$$\begin{aligned} 1 &\le Q^{(2)}(C \setminus F) + |F| - Q^{(2)}(F) \\ &= 1 - {Q^{(2)}}_{c_S} + \frac{1}{2} \sum_{e \in \delta_G(v) \setminus S} x_e \\ &= 1 - {Q^{(2)}}_{c_S} + \frac{2 + \eta_v - q_S - a_S}{2}. \end{aligned}$$

Therefore,

$${Q^{(2)}}_{c_S} \le \frac{2 + \eta_v - q_S - a_S}{2}.$$

The two estimates imply that every nonroot star S centered at v satisfies

$$(66) \qquad {Q^{(2)}}_{c_S} \le \begin{cases} \min\left\{ \dfrac{q_S + a_S}{2}, \dfrac{2 + \eta_v - q_S - a_S}{2} \right\}, & \text{if } v \text{ has even degree in } T, \\ \dfrac{q_S + a_S}{2}, & \text{if } v \text{ has odd degree in } T. \end{cases}$$

These bounds follow from the Blossom inequalities for $Q^{(2)}$, so they hold simultaneously for all stars.

I.4. **A convex combination and the contribution of each star.**

Lemma I.5. *For every nonroot star S,*

$$\bar{e}_S \le \frac{t_S - 1}{t_S + 1}, \qquad \frac{\bar{e}_S}{2} - \frac{1}{4} \le \frac{t_S + 1}{64}.$$

Proof. If $\bar{e}_S = 0$, both inequalities follow from $t_S \ge 1$. Otherwise $e_S > 0$, and greedy DFS together with $\bar{e}_S \le e_S$ and $\bar{e}_S \le 1 - q_S$ gives

$$1 + \bar{e}_S \le 1 + e_S = a_S \le t_S q_S \le t_S(1 - \bar{e}_S).$$

Rearranging proves the first inequality. For the second inequality, use the first to obtain

$$\begin{aligned} \frac{\bar{e}_S}{2} - \frac{1}{4} &\le \frac{t_S - 1}{2(t_S + 1)} - \frac{1}{4} \\ &= \frac{t_S - 3}{4(t_S + 1)}. \end{aligned}$$

Moreover, since $t_S \ge 1$,

$$\begin{aligned} \frac{t_S + 1}{64} - \frac{t_S - 3}{4(t_S + 1)} &= \frac{(t_S + 1)^2 - 16(t_S - 3)}{64(t_S + 1)} \\ &= \frac{(t_S - 7)^2}{64(t_S + 1)} \ge 0. \end{aligned}$$

Combining these inequalities proves $\frac{\bar{e}_S}{2} - \frac{1}{4} \le \frac{(t_S+1)}{64}$. ∎

Let $\bar{d} \in \mathbb{R}^{\mathcal{S}}_+$ be defined by

$$\bar{d}_S := \begin{cases} 0, & S = S_r, \\ \left(\bar{e}_S - \dfrac{q_S}{2} - \dfrac{t_S}{32} \right)_+, & S \ne S_r. \end{cases}$$

Set $Q^{(1)} := Q(\bar{d})$. Its coordinates are

$$Q^{(1)}{}_{\mathrm{rep}(e)} = \begin{cases} \dfrac{1}{3} + \dfrac{2}{3}\left(\bar{e}_S - \dfrac{q_S}{2} - \dfrac{t_S}{32}\right)_+, & e = uv, \quad S = S(u,v) \neq S_r, \\ 1 - \dfrac{2}{3}x_e, & e \notin E(T), \\ \dfrac{1}{3}, & \text{otherwise}, \end{cases}$$

$$Q^{(1)}{}_{c_S} = \begin{cases} 1, & S = S_r, \\ \max\{1/3,\, 1 - \kappa_S(\bar{d})\}, & S \neq S_r. \end{cases}$$

Since $0 \le \bar{d}_S \le \bar{e}_S$ for every nonroot star, Lemma I.2 gives $Q^{(1)} \in \mathcal{P}^\circ$. Set

$$Y := \frac{3}{4}Q^{(1)} + \frac{1}{4}Q^{(2)}.$$

By convexity, $Y \in \mathcal{P}^\circ$.

Recall that

$$\Phi_{G,\mathcal{S}}(Y) = n - 1 + Y(N_H) + \sum_{S \in \mathcal{S}} \max\{Y_{c_S},\, k_S - 1 - Y(S_H)\}.$$

For each nonroot star S, define

$$\chi_S := \max\{(q_S + a_S - 2)_+, \tau_S\}.$$

The next lemma bounds each nonroot star's contribution $\max\{Y_{c_S},\, k_S - 1 - Y(S_H)\}$ to the objective.

Lemma I.6. *Every nonroot star S satisfies*

$$\max\{Y_{c_S},\, k_S - 1 - Y(S_H)\} \le \frac{1}{4} + \frac{q_S + a_S}{8} + \frac{t_S}{64} + \frac{\chi_S}{2}.$$

Moreover, at every nonroot vertex v,

$$\sum_{S \text{ centered at } v} \chi_S \le \eta_v. \tag{67}$$

Proof. We bound both terms in the maximum.

The term $k_S - 1 - Y(S_H)$. Recall that S contains one edge of T and t_S back edges. The edge of T has x-value q_S, and the back edges have total x-value a_S. By the definitions of $Q^{(1)}$ and $Q^{(2)}$,

$$Q^{(1)}(S_H) = \frac{1}{3} + \frac{2}{3}\bar{d}_S + t_S - \frac{2}{3}a_S,$$

$$Q^{(2)}(S_H) = \frac{q_S}{2} + t_S - \frac{a_S}{2}.$$

Since $Y = \frac{3}{4}Q^{(1)} + \frac{1}{4}Q^{(2)}$, we obtain

$$\begin{aligned} Y(S_H) &= \frac{3}{4}\left(\frac{1}{3} + \frac{2}{3}\bar{d}_S + t_S - \frac{2}{3}a_S\right) + \frac{1}{4}\left(\frac{q_S}{2} + t_S - \frac{a_S}{2}\right) \\ &= t_S + \frac{1}{4} + \frac{\bar{d}_S}{2} + \frac{q_S}{8} - \frac{5a_S}{8}. \end{aligned}$$

Using $k_S - 1 = t_S$ gives

$$\begin{aligned} k_S - 1 - Y(S_H) &= -\frac{1}{4} - \frac{\bar{d}_S}{2} - \frac{q_S}{8} + \frac{5a_S}{8} \\ &= \frac{1}{4} + \frac{q_S + a_S}{8} + \frac{a_S - 1 - \bar{d}_S}{2} - \frac{q_S}{4}. \end{aligned}$$

We must therefore bound the last two terms by $t_S/64 + \chi_S/2$. The choice of $\bar{d}_S$ gives

$$\bar{e}_S - \bar{d}_S = \min\left\{\bar{e}_S, \frac{q_S}{2} + \frac{t_S}{32}\right\} \le \frac{q_S}{2} + \frac{t_S}{32}.$$

Since $e_S = (a_S - 1)_+$, $q_S \le 1$, and $\bar{e}_S = \min\{e_S, 1 - q_S\}$, we have

$$e_S - \bar{e}_S = (q_S + a_S - 2)_+.$$

Using $a_S - 1 \le e_S$, we obtain

$$\begin{aligned}\frac{a_S - 1 - \bar{d}_S}{2} - \frac{q_S}{4} &\le \frac{\bar{e}_S - \bar{d}_S}{2} - \frac{q_S}{4} + \frac{(q_S + a_S - 2)_+}{2} \\ &\le \frac{t_S}{64} + \frac{(q_S + a_S - 2)_+}{2} \\ &\le \frac{t_S}{64} + \frac{\chi_S}{2}.\end{aligned}$$

This proves the required bound for $k_S - 1 - Y(S_H)$, including when $e_S = 0$.

We now bound Y_{c_S}. By Lemma I.3, there are two cases for the connector bound in $Q^{(1)}$. We consider each case separately and combine this bound with the corresponding estimate for ${Q^{(2)}}_{c_S}$.

Case 1: ${Q^{(1)}}_{c_S} \le \frac{1}{3} + \frac{2}{3}\tau_S$**.** This case includes every star whose center v has at least two DFS children. If v has exactly one DFS child, it also applies when its parent edge belongs to N or to S_r, or belongs to a nonroot star $S^{\mathrm{par}} = S(u, v)$ satisfying

$$\bar{d}_{S^{\mathrm{par}}} \le o(v) + \tau_S.$$

The connector estimate and the basic bound for $Q^{(2)}$ give

$$\begin{aligned}Y_{c_S} &\le \frac{3}{4}\left(\frac{1}{3} + \frac{2}{3}\tau_S\right) + \frac{1}{4}\frac{q_S + a_S}{2} \\ &= \frac{1}{4} + \frac{q_S + a_S}{8} + \frac{\tau_S}{2} \\ &\le \frac{1}{4} + \frac{q_S + a_S}{8} + \frac{\chi_S}{2}.\end{aligned}$$

Case 2: ${Q^{(1)}}_{c_S} > \frac{1}{3} + \frac{2}{3}\tau_S$**.** Write $S = S(v, w)$, where w is the only DFS child of its center v. Its parent edge uv belongs to a nonroot star

$$S^{\mathrm{par}} := S(u, v), \qquad u := \mathrm{par}(v),$$

and $\bar{d}_{S^{\mathrm{par}}} - o(v) > \tau_S \ge 0$. We bound Y_{c_S} using the increase chosen at S^{par}. Lemma I.3 gives

$${Q^{(1)}}_{c_S} \le \frac{1}{3} + \frac{2}{3}(\bar{d}_{S^{\mathrm{par}}} - o(v)).$$

The center v has one parent and one child, so it has degree two in T. Using the stronger bound for $Q^{(2)}$, we obtain

$$\begin{aligned}Y_{c_S} &\le \frac{3}{4}\left(\frac{1}{3} + \frac{2}{3}(\bar{d}_{S^{\mathrm{par}}} - o(v))\right) + \frac{1}{4}\left(\frac{2 + \eta_v - q_S - a_S}{2}\right) \\ &= \frac{1}{4} + \frac{q_S + a_S}{8} + \frac{\bar{d}_{S^{\mathrm{par}}} - o(v)}{2} - \frac{q_S + a_S - 1 - \eta_v/2}{4}.\end{aligned} \tag{68}$$

It remains to bound the last two terms by $t_S/64$.

Since $q_{S^{\mathrm{par}}} = x_{uv}$, the degree equation at v is

$$q_{S^{\mathrm{par}}} + q_S + a_S + o(v) = 2 + \eta_v.$$

Thus

$$q_S + a_S - 1 - \frac{\eta_v}{2} = 1 + \frac{\eta_v}{2} - q_{S^{\mathrm{par}}} - o(v).$$

Also, $\bar{d}_{S^{\mathrm{par}}} > o(v) \geq 0$, so S^{par} is positive and

$$\bar{d}_{S^{\mathrm{par}}} = \bar{e}_{S^{\mathrm{par}}} - \frac{q_{S^{\mathrm{par}}}}{2} - \frac{t_{S^{\mathrm{par}}}}{32}.$$

Substituting these identities gives

$$\begin{aligned}
&\frac{\bar{d}_{S^{\mathrm{par}}} - o(v)}{2} - \frac{q_S + a_S - 1 - \eta_v/2}{4} \\
&\quad = \frac{\bar{e}_{S^{\mathrm{par}}}}{2} - \frac{1}{4} - \frac{t_{S^{\mathrm{par}}}}{64} - \frac{o(v)}{4} - \frac{\eta_v}{8} \\
&\quad \leq \frac{\bar{e}_{S^{\mathrm{par}}}}{2} - \frac{1}{4} - \frac{t_{S^{\mathrm{par}}}}{64}.
\end{aligned}$$

Lemma I.5 now gives

$$\frac{\bar{e}_{S^{\mathrm{par}}}}{2} - \frac{1}{4} - \frac{t_{S^{\mathrm{par}}}}{64} \leq \frac{1}{64} \leq \frac{t_S}{64}.$$

The last inequality uses $t_S \geq 1$; the successor S need not be positive. This proves the connector bound in Case 2. Together, the two cases and the bound for the representative edges prove the first part of the lemma. **The sum of the additional terms at a center v.** We now prove (67). Fix a nonroot vertex v. We want to prove

$$\sum_{S \text{ centered at } v} \chi_S \leq \eta_v.$$

Recall that

$$\begin{aligned}
x(\delta_G(v)) &= 2 + \eta_v, \\
\tau_S &= \big(1 - \max\{0, 2 - q_S - a_S, q_S + a_S - \eta_v\}\big)_+, \\
\chi_S &= \max\{(q_S + a_S - 2)_+, \tau_S\}.
\end{aligned}$$

For every star S centered at v, we have

$$q_S + a_S = x(S) \leq x(\delta_G(v)) = 2 + \eta_v,$$

and hence $(q_S + a_S - 2)_+ \leq \eta_v$.

To bound τ_S, suppose first that $\tau_S > 0$. We obtain the following inequalities from the definition of τ_S:

$$\begin{aligned}
\tau_S &\leq 1 - (2 - q_S - a_S) = q_S + a_S - 1, \\
\tau_S &\leq 1 - (q_S + a_S - \eta_v) = 1 + \eta_v - q_S - a_S.
\end{aligned}$$

Adding gives $2\tau_S \leq \eta_v$. This inequality also holds when $\tau_S = 0$. Thus both terms defining χ_S are at most η_v, and

$$\chi_S \leq \eta_v.$$

Case 1: At most one star centered at v has $\chi_S > 0$. All other terms in the sum are zero, so the individual bound above gives

$$\sum_{S \text{ centered at } v} \chi_S \leq \eta_v.$$

Case 2: At least two stars centered at v have $\chi_S > 0$. We show that each positive term satisfies

$$\chi_S \leq q_S + a_S - 1.$$

Consider a star S with $\chi_S > 0$. We must have $q_S + a_S > 1$, since otherwise both terms in

$$\chi_S = \max\{(q_S + a_S - 2)_+, \tau_S\}$$

would be zero. Therefore,

$$(q_S + a_S - 2)_+ \le q_S + a_S - 1.$$

The other term satisfies the same bound: when $\tau_S > 0$, we proved above that $\tau_S \le q_S + a_S - 1$; when $\tau_S = 0$, the inequality follows from $q_S + a_S > 1$. Thus $\chi_S \le q_S + a_S - 1$.

We now sum over the stars with $\chi_S > 0$. Their edge sets are disjoint and contained in $\delta_G(v)$, so

$$\sum_{\substack{S \text{ centered at } v \\ \chi_S > 0}} (q_S + a_S) \le x(\delta_G(v)) = 2 + \eta_v.$$

There are at least two such stars. Consequently,

$$\begin{aligned}
\sum_{S \text{ centered at } v} \chi_S &\le \sum_{\substack{S \text{ centered at } v \\ \chi_S > 0}} (q_S + a_S - 1) \\
&= \sum_{\substack{S \text{ centered at } v \\ \chi_S > 0}} (q_S + a_S) - \sum_{\substack{S \text{ centered at } v \\ \chi_S > 0}} 1 \\
&\le (2 + \eta_v) - 2 \\
&= \eta_v.
\end{aligned}$$

This proves (67). ■

I.5. **The resulting tour bound.** Put $L := \textsc{SubtourLP}(G)$. We now sum the bounds from Lemma I.6. The additional terms from the nonroot stars satisfy

$$\begin{aligned}
\sum_{S \neq S_r} \left(\frac{t_S}{64} + \frac{\chi_S}{2} \right) &\le \frac{n-4}{64} + \frac{1}{2} \sum_{v \neq r} \eta_v \\
&= \frac{n-4}{64} + \Delta - \frac{\eta_r}{2},
\end{aligned}$$

where we used (59) and $\sum_v \eta_v = 2\Delta$.

For every $e \in N$, we have $Y_{\mathrm{rep}(e)} = 1/4 + x_e/8$. The edges in N and the nonroot stars partition $E \setminus S_r$. Since the root has one DFS child, $q_{S_r} + a_{S_r} = 2 + \eta_r$, and thus

$$|N| + |\mathcal{S}| - 1 = n - 2, \qquad x(E \setminus S_r) = L - 2 - \eta_r.$$

Consequently,

$$\begin{aligned}
Y(N_H) + \sum_{S \neq S_r} \left(\frac{1}{4} + \frac{q_S + a_S}{8} \right) &= \frac{n-2}{4} + \frac{L - 2 - \eta_r}{8} \\
&= \frac{3}{8}(n-2) + \frac{\Delta}{8} - \frac{\eta_r}{8}.
\end{aligned}$$

It remains to include the root star. Its connector has value one in both $Q^{(1)}$ and $Q^{(2)}$, and the coordinate of its tree edge is unchanged in $Q^{(1)}$. Substituting $a_{S_r} = 2 + \eta_r - q_{S_r}$ gives

$$\begin{aligned}
k_{S_r} - 1 - Y((S_r)_H) &= -\frac{1}{4} - \frac{q_{S_r}}{8} + \frac{5 a_{S_r}}{8} \\
&= 1 + \frac{5}{8}\eta_r - \frac{3}{4} q_{S_r}.
\end{aligned}$$

Its contribution is therefore

$$\begin{aligned}
\max\{Y_{c_{S_r}},\, k_{S_r} - 1 - Y((S_r)_H)\} &= 1 + \left(\frac{5}{8}\eta_r - \frac{3}{4} q_{S_r} \right)_+ \\
&\le 1 + \frac{5}{8}\eta_r.
\end{aligned}$$

Using the formula for $\Phi_{G,\mathcal{S}}(Y)$, we substitute the bound for the root star and the bounds for the nonroot stars, and then apply the estimates for their sums:

$$\begin{aligned}
\Phi_{G,\mathcal{S}}(Y) &= n - 1 + Y(N_H) + \sum_{S\in\mathcal{S}} \max\{Y_{c_S},\, k_S - 1 - Y(S_H)\} \\
&\le n + \frac{5}{8}\eta_r + Y(N_H) + \sum_{S\neq S_r} \max\{Y_{c_S},\, k_S - 1 - Y(S_H)\} \\
&\le n + \frac{5}{8}\eta_r + Y(N_H) + \sum_{S\neq S_r} \left(\frac{1}{4} + \frac{q_S + a_S}{8} + \frac{t_S}{64} + \frac{\chi_S}{2}\right) \\
&= \frac{11}{8}n - \frac{3}{4} + \frac{\Delta}{8} + \frac{\eta_r}{2} + \frac{1}{64}\sum_{S\neq S_r} t_S + \frac{1}{2}\sum_{S\neq S_r} \chi_S \\
&\le \frac{11}{8}n - \frac{3}{4} + \frac{\Delta}{8} + \frac{\eta_r}{2} + \frac{n-4}{64} + \frac{1}{2}\sum_{S\neq S_r} \chi_S \\
&\le \frac{11}{8}n - \frac{3}{4} + \frac{\Delta}{8} + \frac{\eta_r}{2} + \frac{n-4}{64} + \frac{1}{2}\sum_{v\neq r} \eta_v \\
&= \frac{11}{8}n - \frac{3}{4} + \frac{\Delta}{8} + \frac{\eta_r}{2} + \frac{n-4}{64} + \frac{2\Delta - \eta_r}{2} \\
&= \frac{11}{8}n - \frac{3}{4} + \frac{9}{8}\Delta + \frac{n-4}{64} \\
&= \frac{89}{64}n + \frac{9}{8}\Delta - \frac{13}{16}.
\end{aligned}$$

Since $Y \in \mathcal{P}^\circ$ and $L = n + \Delta$, we obtain

$$\begin{aligned}
\mathrm{tsp}(G) &\le \textsc{StarReconnectionLP}_{\mathrm{OPT}}(G,\mathcal{S}) \\
&\le \Phi_{G,\mathcal{S}}(Y) \\
&\le \frac{89}{64}n + \frac{9}{8}\Delta - \frac{13}{16} \\
&= \frac{89}{64}L - \frac{17}{64}\Delta - \frac{13}{16} \\
&\le \frac{89}{64}L.
\end{aligned}$$

I.6. **Specialization to $L = n$.**

Proposition I.7 (Specialization to $L = n$)**.** *Assume $L = n$. Then $\eta_v = 0$ for every vertex and $\bar{e}_S = e_S$ for every nonroot star. For every vector d satisfying $d_{S_r} = 0$ and $0 \le d_S \le e_S$, the point $Q(d)$ constructed in this appendix agrees exactly with (9) and (10), and belongs to $\mathcal{P}^\circ$. Its objective value is*

$$\Phi_{G,\mathcal{S}}(Q(d)) = \frac{4}{3}n - \frac{2}{3} + \sum_{S\neq S_r} \max\left\{Q(d)_{c_S} - \frac{1}{3},\, \frac{2}{3}(e_S - d_S)\right\}. \tag{69}$$

The point $Q^{(2)}$ satisfies the connector bounds in (12). The general choice of $\bar{d}$ reduces to

$$\bar{d}_{S_r} = 0, \qquad \bar{d}_S = \left(e_S - \frac{q_S}{2} - \frac{t_S}{32}\right)_+ \quad (S \neq S_r).$$

With $Q^{(1)} = Q(\bar{d})$ and $Y = \frac{3}{4}Q^{(1)} + \frac{1}{4}Q^{(2)}$, we have $Y \in \mathcal{P}^\circ$ and, for every nonroot star S,

$$\max\{Y_{c_S},\, k_S - 1 - Y(S_H)\} \le \frac{1}{4} + \frac{q_S + a_S}{8} + \frac{t_S}{64}. \tag{70}$$

Consequently,

$$\Phi_{G,\mathcal{S}}(Y) \le \frac{89n - 52}{64}, \tag{71}$$

and StarMatch *returns a tour satisfying this bound.*

Proof. Since $L = n$, we have $\eta_v = 0$ for every vertex. Thus every singleton cut is tight. For a nonroot star $S = S(u, v)$, we obtain $q_S + a_S \le x(\delta_G(u)) = 2$. Together with $q_S \le 1$, this gives $e_S \le 1 - q_S$, and hence $\bar{e}_S = e_S$. We can therefore replace $\bar{e}_S$ by e_S in Lemma I.5 and in the definition of $\bar{d}$. The admissibility conditions are exactly those in Section 6.2, and Lemma I.2 shows that $Q(d) \in \mathcal{P}^\circ$.

We next check that the two constructions give the same point $Q(d)$. For every nonroot star S' centered at v, we have $\eta_v = 0$ and hence $\tau_{S'} = 0$. The equality case proved after Lemma I.3 then gives precisely the nonroot connector values in (10). The root connector has value one in both constructions, and the values on the nonconnector edges agree with (9).

For every nonroot star S, the definition of $Q(d)$ gives

$$Q(d)(S_H) = \frac{1}{3} + \frac{2}{3}d_S + t_S - \frac{2}{3}a_S.$$

When $a_S > 1$, we have $e_S = a_S - 1$. When $a_S \le 1$, admissibility gives $d_S = 0$, and $Q(d)_{c_S} \ge 1/3$. Thus replacing $a_S - 1$ by e_S in the maximum defining the contribution of S does not change its value. The root star contributes one, every edge in N_H has value $1/3$, and $|N| + |\mathcal{S}| = n - 1$. Substituting into (58) gives (69). Setting $d = 0$ also gives (8).

Setting $\eta_v = 0$ in (66) gives the connector bounds in (12). Also, $q_S + a_S \le 2$ and $\tau_S = 0$ imply $\chi_S = 0$ for every nonroot star. Lemma I.6 therefore gives (70). Finally, applying the calculation in Appendix I.5 with $\Delta = \eta_r = 0$ gives (71). Corollary C.2 then gives a tour satisfying this bound. ■

Appendix J. Further improvements

We describe two refinements of the StarReconnectionLP analysis that can improve the upper bound on the integrality ratio of the subtour LP for graphic TSP: a third point Z obtained from a circulation and a sharper bound on the number of back edges. We give the constructions and the resulting estimates, but do not present their proofs or their combined use in the approximation analysis in full. Throughout this section, we use the setup of Appendix I with $L = n$, as specialized in Proposition I.7. In this case, the input extreme solution x^* already satisfies $x^*_e \le 1$, so we choose $x = x^*$ in StarMatch.

J.1. **The point Z and its circulation.** In the Mömke–Svensson circulation of Section E.2, the first chosen back edge entering a nonroot star is free, and each additional one has unit cost. The StarReconnectionLP allows the second chosen back edge to be free as well, provided that a chosen back edge starts at the child of the star's center.

More precisely, let $S = S(u, v)$ be a nonroot star. If at least two chosen back edges belong to S and a chosen back edge starts at v, we can increase $Z_{\mathrm{rep}(uv)}$ from $1/3$ to 1 while preserving feasibility. This decreases $k_S - 1 - Z(S_H)$ by $2/3$, which pays for the second chosen back edge in S.

We therefore use the following costs. At each nonroot star, the first incoming unit is free. A second unit is free if it continues along a back edge starting at v; all further incoming units have cost one. At the root star, the first two incoming units are free. As in the Mömke–Svensson construction, we require at least one unit of flow through the arcs corresponding to every nonroot connector and every edge of N_H.

Figure 30 gives a circulation formulation of these costs. Back arcs belonging to S enter h_S, while back arcs starting at v leave t_v. The first incoming unit can return through $h_S \to z_S$ at cost zero. A second unit can use $h_S \to t_v$ at cost zero if it continues along a back arc starting at v. Each of these arcs has upper bound one, and the remaining flow returns through $h_S \to u$ at unit cost. If no chosen back edge covers c_S, the cycle $u \to z_S \to h_S \to u$ satisfies its lower bound at cost one. We denote the total circulation cost by $C^Z(f)$.

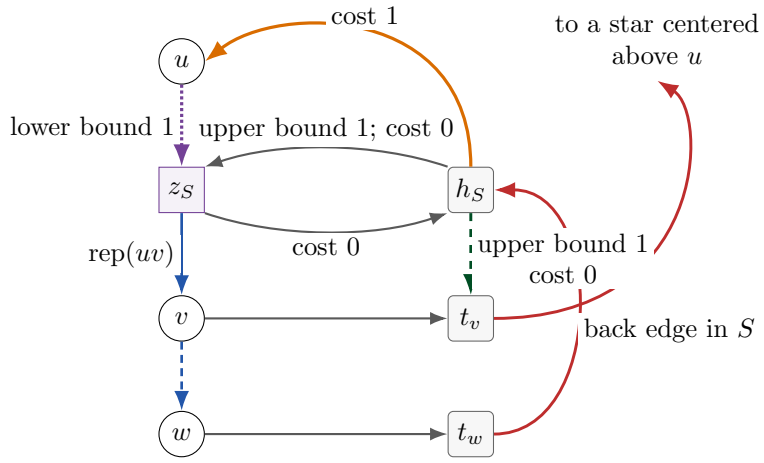


Figure 30. The gadget at $S = S(u, v)$. The green arc permits the second free unit. Unlabelled arcs have cost zero and no upper bound; the dashed blue segment denotes a downward path.

Choose an optimal integral circulation f, and let B be the set of back edges whose arcs carry positive flow.

The edges $\mathrm{rep}(e)$ with $e \in E(T)$ and the connectors form a tree in H°. For each $b \in B$, consider the cycle formed by $\mathrm{rep}(b)$ and the path joining its endpoints in this tree. Assign value $1/3$ to every edge belonging to at least one of these cycles, and value one to all remaining edges. Thus the root connector has value one, and the flow lower bounds ensure that every edge of N_H has value $1/3$.

For each nonroot star $S = S(u, v)$ with $|B \cap I(u, v)| \geq 2$ and one unit of flow on $h_S \to t_v$, increase $Z_{\mathrm{rep}(uv)}$ to one. Flow conservation at t_v implies that a back edge in B starts at v, so this increase satisfies the condition described above.

The resulting point belongs to $\mathcal{P}^\circ$ and satisfies

$$\Phi_{G,\mathcal{S}}(Z) \leq \frac{4n-2}{3} + \frac{2}{3}C^Z(f). \tag{72}$$

J.2. **Laminarity and unpartitionable vertices.** We call a vertex v partitionable if there is a set $F \subseteq \delta_G(v)$ with $x(F) = 1$, and unpartitionable otherwise. Since $x(\delta_G(v)) = 2$, a partitionable vertex has its incident edges partitioned into two sets of weight one. Let b be the number of nonroot unpartitionable vertices.

The uncrossing argument of [10] gives a laminar family $\mathcal{L}$ of tight cut sets whose cut vectors span all tight-cut vectors on the support of x. We choose these sets to exclude r and include every nonroot singleton and $V \setminus \{r\}$.

In the inclusion tree of $\mathcal{L}$, the parent of a set is the smallest set in the family that properly contains it. The root is $V \setminus \{r\}$, the leaves are the $n-1$ nonroot singletons, and the children of each nonsingleton set partition that set.

An unpartitionable singleton $\{v\}$ cannot have a parent with exactly two children. Indeed, if the other child is A, tightness of $\{v\}$, A, and $A \cup \{v\}$ gives

$$2x(E_G(\{v\}, A)) = x(\delta_G(v)) + x(\delta_G(A)) - x(\delta_G(A \cup \{v\})) = 2.$$

This would give a set of edges incident to v with total weight one, contradicting unpartitionability.

Thus each of the b unpartitionable singletons has a parent with at least three children. Counting the nodes of the inclusion tree gives at most $2n - 3 - \lceil b/3 \rceil$ sets. Since x is positive and extreme, its tight-cut vectors have rank m, and hence $\mathcal{L}$ contains at least m sets. Therefore,

$$m \leq 2n - 3 - \left\lceil \frac{b}{3} \right\rceil. \tag{73}$$

Every back edge belongs to one star, and the root star contains at least two back edges. Consequently,

$$\sum_{S \neq S_r} t_S \leq n - 4 - \left\lceil \frac{b}{3} \right\rceil. \tag{74}$$

This improves the count used when summing the local star bounds. Partitionable centers also give a stronger local estimate: for a positive star S centered at such a vertex, greedy DFS gives

$$e_S \leq \left\lfloor \frac{t_S - 1}{2} \right\rfloor q_S.$$

Thus partitionable centers improve the local estimates, while unpartitionable vertices give a saving in the total number of back edges.

AI methodology

The authors developed the StarDeletionLP and StarReconnectionLP frameworks and their application to k-vertex-connected graphs of maximum degree at most $k+1$.

For 4-regular, 4-vertex-connected graphs, the authors developed the initial vector, the procedure for pairing demands, and the use of shared corrections, where increasing one edge satisfies two

demands. The counting arguments and computations leading to the final bound on paid increases in Section G were developed by the authors with assistance from OpenAI's GPT-5.6 Pro.

For general graphic TSP, the authors developed the constructions of $Q(d)$ and $Q^{(2)}$, including the idea of increasing the coordinates of edges in removable stars. GPT-6 Astra assisted in refining the choice of the increase parameters d and deriving the bounds obtained from convex combinations of these points in Appendix I. This assistance included developing and checking the inequalities used to bound the contribution of each star and combining these estimates to obtain the final tour bounds.

AI tools were used in writing this manuscript, including drafting and rewriting substantial portions of the text. All the text has been reread and edited by the authors. AI tools also assisted with checking and simplifying proofs, as well as developing Lean formalizations. The authors take full responsibility for the mathematical claims and proofs in this work.