ON SOME k-FOLD GENERALIZATIONS OF LOVÁSZ THETA AND THEIR SANDWICH THEOREMS

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Abstract. We study several k-fold generalizations of the Lovász theta function associated with the maximum k-colorable induced subgraph problem. The first is the Narasimhan–Manber parameter ϑ_k. We prove that, for graphs whose adjacency matrix belongs to a homogeneous partially coherent algebra, this parameter is recovered by the theta number of the Cartesian product with the complete graph on k vertices. This class includes distance-regular and 1-walk-regular graphs, and thus our result generalizes a theorem by Sinjorgo and Sotirov (2022) for graphs that are vertex- and edge-transitive. We introduce a new parameter φ_k obtained from orthonormal representations of graphs and show the inequality $\varphi_k \leq \vartheta_k$. For both parameters, we study the smallest k for which the parameter is equal to the number of vertices; these saturation parameters yield lower bounds on the chromatic number. We determine which vertex-weighted versions of these parameters are gauges, and discuss a natural definition for the k-fold theta body of a graph. We conclude with open questions comparing ϑ_k, φ_k, $\vartheta(G \,\square\, K_k)$, and related convexifications.

1. Introduction

The Lovász theta number $\vartheta(G)$ of a graph G, introduced in [17], has remarkable applications across combinatorial optimization and spectral graph theory [8, 11, 12]. Perhaps one of its most famous properties is the sandwich theorem (see [14]), which states that

$$\alpha(G) \leq \vartheta(G) \leq \chi_f(\overline{G}) \leq \chi(\overline{G}), \tag{1}$$

where α denotes the independence number, χ_f is the fractional chromatic number, χ is the chromatic number, and $\overline{G}$ is the complement of G. Moreover, $\vartheta(G)$ can be formulated as the optimal value of a semidefinite program (SDP) which allows it to be computed in polynomial time to arbitrary precision.

Let G be a graph and let $k \geq 1$. We denote by $\alpha_k(G)$ the largest number of vertices of an induced subgraph of G that is k-colorable. The k-multicoloring number $\chi_k(G)$ of G, sometimes called [4] the k-fold colouring number of G, is the smallest number of colors needed so that one can assign to each vertex a k-subset of these colors and adjacent vertices are assigned disjoint sets. Thus, $\alpha_1 = \alpha$ and $\chi_1 = \chi$. We informally refer to such parameters as *k-fold generalizations*. The main topic of this paper is a k-fold generalization of ϑ, denoted ϑ_k and introduced by Narasimhan and Manber [21],

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which recovers ϑ when $k = 1$ and satisfies the following sandwich theorem:

$$\alpha_k(G) \leq \vartheta_k(G) \leq \chi_k(\overline{G}). \tag{2}$$

Moreover, $\vartheta_k(G)$ also admits a formulation as the optimal value of an SDP; see, e.g., [1, Sec. 5.3].

A key motivation for this paper and other recent work on ϑ_k is to determine the quality of approximations of α_k via geometric strengthenings of ϑ_k, and subsequently the quality of bounds for χ that can be obtained from ϑ_k. We survey below some of the results that motivate this paper.

Let G be a graph on n vertices. Mohar and Poljak [19] pointed out that if $k \geq \chi(G)$, then $n \leq \alpha_k(G)$, whence $n \leq \vartheta_k(G)$. This elementary remark shows that ϑ_k can be used to obtain lower bounds on χ; namely,

$$\mu(G) \coloneqq \min\{\, k : \vartheta_k(G) = n\} \leq \chi(G). \tag{3}$$

The recent work [6] has shown a strong generalization of this result by exploring vertex weights and by replacing $\chi(G)$ by $\chi_f(G)$ in (3). In this paper we will introduce an SDP formulation for $\mu(G)$, study its behavior in highly regular graphs, and present a conjecture relating it to $\vartheta(\overline{G})$.

Recent works [2, 12] provide other frameworks for converting upper bounds for α into lower bounds for χ. In the approach of [12], the input graph parameter is applied to the Cartesian product $G \,\square\, K_k$. Motivated by this, Sinjorgo and Sotirov [25] compared $\vartheta(G \,\square\, K_k)$ and $\vartheta_k(G)$ in order to determine which of ϑ or ϑ_k yields a tighter bound for χ. Numerical results from [15] led them to propose the following conjecture:

Conjecture 1 ([25, Conjecture 1])**.** For any graph G and integer number $k \geq 1$, one has $\vartheta(G \square K_k) \leq \vartheta_k(G)$. Equality holds if $\vartheta_k(G) = k\vartheta(G)$.

Sinjorgo and Sotirov provided further support for the conjecture by proving the following result and by computing the (common) values of these two parameters for various classical algebraic families of graphs.

Theorem 2 ([25, Theorem 5.11])**.** Let G be a graph with n vertices that is both vertex- and edge-transitive. Let $k \geq 1$ be an integer. Then $\vartheta(G \,\square\, K_k) = \min\{k\vartheta(G), n\} = \vartheta_k(G)$.

We will extend this result to a much larger class of graphs that include distance-regular and 1-walk-regular graphs.

The paper [15] presents several lifted formulations that attempt to approximate ϑ_k. We identify that one of them suggests the definition of k-fold theta body, which we formalize in this paper. This led us to study whether weighted ϑ_k is a monotone gauge (it is not), and to introduce several variants based on the standard theory for $k = 1$. We study their behavior with regards to convexity and their relation to χ and χ_f. One of the objects we introduce has an interesting connection to the minimum dimension of orthonormal representations. We also prove several inequalities, including examples that show some of them are strict.

All the works mentioned above consider only integer values of k. A subtle point of this work is that allowing fractional/real values of k can lead to simpler and more natural statements, smoothing out the rounding phenomena that arise when k is restricted to integers.

Our contributions

Our main contributions are the substantial extension of Theorem 2, and the introduction of the parameter $\varphi_k(G, w)$, along with the proof of several of its properties.

- In Section 2 we recall the definition of a homogeneous partially coherent algebra, and discuss several consequences that it implies for the optimal solutions determining ϑ. In Section 2.1, we generalize Theorem 2 to the class of graphs whose adjacency matrix belongs to these algebras, which includes vertex-transitive graphs, distance-regular graphs (and thus strongly regular graphs), and 1-walk-regular graphs. This represents a wide extension of graphs for which Conjecture 1 is known to hold, and it provides a unified approach to several families of graphs considered in [25, Sections 5 and 6]. In Section 2.2, we show that $\mu(G) = \vartheta(\overline{G})$ for graphs whose adjacency matrix defines a homogeneous partially coherent algebra, and we propose the conjecture that $\mu(G) \leq \vartheta(\overline{G})$ for all graphs.

- In Section 3, we define and study the graph parameter φ_k inspired by one of the many alternative formulations of ϑ given by Lovász [17], based on orthonormal representations. We show that, although $\varphi_1 = \vartheta_1$, for other values of k one only has the inequality $\varphi_k(G) \leq \vartheta_k(G)$. We relate φ_k to the minimum dimension $\xi(\overline{G})$ of an orthonormal representation of $\overline{G}$: we show that for $k := \xi(\overline{G})$ one has $\varphi_k(G) = n$, the number of vertices of G.
- In Section 4, we introduce $\alpha'_k(G)$ to be the largest number of vertices in an induced subgraph of G whose fractional chromatic number is $\leq k$. We show that $\alpha'_k(G) \leq \vartheta_k(G)$, so that ϑ_k cannot distinguish k-colorability from having *fractional* chromatic number bounded by k.
- In Section 5, we extend several parameters to vertex-weighted versions, preserving the known inequalities. We show that, if $w \in \mathbb{R}^V_+$ is a vector of nonnegative vertex weights, the weighted variation $\vartheta_k(G, w)$ of $\vartheta_k(G)$ is **not** sublinear, while $\varphi_k(G, w)$ is. In particular, unlike the $k = 1$ case, $\vartheta_k(G, w)$ cannot in general be described as the support function of a convex body analogous to the theta body.
- In Section 5.4, we recall the vertex-weighted version of the ψ_k studied by Lovász in the context of k-perfect graphs, and we show that $\varphi_k(G, w) \leq \psi_k(\overline{G}, w)$. Thus, φ_k is a variant that satisfies a sandwich inequality conjectured by Narasimhan [20, Sec. 4.4.3] for ϑ_k.
- In Section 6, we introduce
$$\mathrm{TH}_k(G) := k\,\mathrm{TH}(G) \cap [0, 1]^V$$
and its support function $\theta_k(G, w)$. We give a semidefinite representation for $\mathrm{TH}_k(G)$, prove $\alpha_k(G, w) \leq \varphi_k(G, w) \leq \theta_k(G, w)$, and discuss how this construction relates to matrix-lifting relaxations of [15].

Notation and background on ϑ_k

We typically denote a graph as $G = (V, E)$, with $n := |V|$. We write $\mathbb{S}^V$ for the set of real symmetric $V \times V$ matrices, and $\mathbb{S}^V_+$ for the subset of these which are positive semidefinite. If $M \in \mathbb{S}^V$, then we shall typically order its eigenvalues as
$$\lambda_1(M) \geq \cdots \geq \lambda_n(M).$$
The Schur product of matrices A and B of same order is defined by
$$(A \circ B)_{ij} = A_{ij} B_{ij}.$$
The (trace) inner product of $A, B \in \mathbb{S}^V$ is $\langle A, B\rangle = \mathrm{Tr}(AB)$. It is convenient to observe that
$$\langle A, B\rangle = \langle J, A \circ B\rangle\,,$$
where $J = \mathbb{1}\mathbb{1}^\mathsf{T}$ is the matrix of all-ones and $\mathbb{1}$ is the vector of all-ones. That is, $\langle A, B\rangle$ is the sum of all entries of the matrix $A \circ B$. The operator diag maps a $V \times V$ matrix Y to a vector in $\mathbb{R}^V$ consisting of the diagonal entries of Y.

For each $k \in [n] := \{1, \ldots, n\}$ and $M \in \mathbb{S}^V$, denote the Ky Fan k-sum
$$\Lambda_k(M) := \sum_{i=1}^{k} \lambda_i(M).$$

Recall that $\Lambda_k(M)$ is the optimal value of the following primal-dual pair of SDPs (see [1, 7, 22]):

$$\begin{aligned}\Lambda_k(M) &= \max\{\, \langle M, X\rangle \,:\, 0 \preccurlyeq X \preccurlyeq I,\, \langle I, X\rangle = k\} \\ &= \min\{\, k\nu + \langle I, Y\rangle \,:\, \nu \in \mathbb{R},\, Y \in \mathbb{S}^V_+,\, M \preccurlyeq Y + \nu I\};\end{aligned} \tag{4}$$

here we write $A \preccurlyeq B$ to mean that $B - A$ is positive semidefinite. Using these formulations, Λ_k may be defined for *real* nonnegative k.

Denote the adjacency matrix of a graph G as A_G. For each $k \in [n]$, the graph parameter ϑ_k is defined by

$$\vartheta_k(G) \coloneqq \max\{\, \langle J, X\rangle \,:\, \langle I, X\rangle = k,\, A_G \circ X = 0,\, 0 \preccurlyeq X \preccurlyeq I\}. \tag{5}$$

When $k = 1$, the constraint $X \preccurlyeq I$ is implied by the constraints $X \succcurlyeq 0$ and $\langle I, X\rangle = 1$, so $\vartheta_1(G)$ recovers the famous Lovász theta number $\vartheta(G)$:

$$\vartheta(G) = \max\{\, \langle J, X\rangle \,:\, \langle I, X\rangle = 1,\, A_G \circ X = 0,\, X \succcurlyeq 0\}. \tag{6}$$

Note that the SDP in (5) is perfectly well defined for each k in the *real* interval $[1, n]$. We adopt this extended definition of ϑ_k. Furthermore, the optimal value in (5) does not change if one replaces the constraint $\langle I, X\rangle = k$ with $\langle I, X\rangle \le k$, which shows that $\vartheta_k(G)$ is a nondecreasing function of k.

Strong duality holds (see [1] or [15, Section 3.1]), and therefore

$$\vartheta_k(G) = \min\left\{\, k\nu + \langle I, Y\rangle : \nu \in \mathbb{R},\, Y \in \mathbb{S}^V_+,\, Z \in \mathbb{S}^V,\, Z \circ A_G = Z,\, \nu I + Y + Z \succcurlyeq J \,\right\}.$$

2. ϑ_k for highly regular graphs

In this section we explore two aspects of ϑ_k for graphs admitting a strong form of regularity, which we introduce below. For these graphs, we significantly extend a result of Sinjorgo and Sotirov [25] comparing $\vartheta_k(G)$ and $\vartheta(G \,\square\, K_k)$; and we study the chromatic number lower bound $\mu(G)$ defined in (3).

We follow a definition from [18]: for a graph G with adjacency matrix A_G, we say that a subalgebra $\mathcal{A}_G$ of $\mathbb{R}^{V\times V}$ is a partially coherent algebra with respect to I and A_G if it contains I, J and A_G, is self-adjoint, and is closed under Schur product with I and A_G. We say that $\mathcal{A}_G$ is homogeneous if every matrix in it has constant diagonal.

The following are all examples of graphs whose adjacency matrix belongs to a homogeneous partially coherent algebra: distance-regular graphs, vertex-transitive graphs, or more generally graphs whose adjacency matrix belongs to homogeneous coherent algebras; and 1-walk-regular graphs. In [5], some sufficient conditions for $\vartheta(G)\vartheta(\overline{G}) = n$ were introduced, and they are satisfied by graphs whose adjacency matrix belongs to a homogeneous partially coherent algebra. We summarize the main result we need below.

Theorem 3 (Corollary 4.2 in [5].)**.** Let G be a graph on n vertices whose adjacency matrix belongs to a homogeneous partially coherent algebra. Then

$$\vartheta(G)\vartheta(\overline{G}) = n.$$

We may also obtain important information about the largest eigenvalue of optimal solutions.

Lemma 4. Let G be a graph on n vertices. If there exists an optimal solution $\hat{X}$ for $\vartheta \coloneqq \vartheta(G)$ in (6) with constant diagonal, then

$$\hat{X} \preccurlyeq \frac{\vartheta}{n} I, \tag{7}$$

and

$$\hat{X}\mathbb{1} = \frac{\vartheta}{n}\mathbb{1}. \tag{8}$$

Moreover, if the adjacency matrix of G belongs to a homogeneous partially coherent algebra, then such solution exists.

Proof. Let $\hat{X}$ be an optimal solution for (6) with constant diagonal. Let h be a λ-eigenvector of $\hat{X}$ such that $\|h\| = 1$. Define $\hat{Y} \coloneqq n(\hat{X} \circ hh^{\mathsf{T}}) \succcurlyeq 0$. We show that $\hat{Y}$ is feasible in (6). If ij is an edge, then $\hat{Y}_{ij} = nh_ih_j\hat{X}_{ij} = 0$. Since $\hat{X}$ has constant diagonal, we have $\operatorname{diag}(\hat{X}) = \frac{1}{n}\mathbb{1}$, so $\operatorname{diag}(\hat{Y}) = h \circ h$. Thus, $\operatorname{Tr}(\hat{Y}) = \|h\|^2 = 1$. Now $\hat{X}$ is optimal in (6), so a comparison of the objective values of $\hat{X}$ and $\hat{Y}$ yields

$$\vartheta = \langle J, \hat{X} \rangle \geq \langle J, \hat{Y} \rangle = n\langle J, \hat{X} \circ hh^{\mathsf{T}} \rangle = n\langle \hat{X}, hh^{\mathsf{T}} \rangle = nh^{\mathsf{T}}\hat{X}h = n\lambda.$$

Thus, (7) is proved. Moreover, the Rayleigh quotient characterization of the largest eigenvalue of $\hat{X}$ gives

$$\vartheta = \langle J, \hat{X} \rangle \leq n\ \lambda_{\max}(\hat{X}) \leq n\ \frac{\vartheta}{n} = \vartheta,$$

thus $\mathbb{1}$ is an eigenvector for the eigenvalue $\lambda_{\max}(\hat{X}) = \vartheta/n$, proving (8).

For the last part, take any optimal solution X' in (6), and let $\hat{X}$ be its orthogonal projection onto the homogeneous partially coherent algebra $\mathcal{A}_G$. Since $J, I \in \mathcal{A}_G$, we have $\langle J, X' \rangle = \langle J, \hat{X} \rangle$ and $\langle I, X' \rangle = \langle I, \hat{X} \rangle$. That $\hat{X} \succcurlyeq 0$ follows from the standard fact that the projection of a positive semidefinite matrix onto a $*$-algebra is positive semidefinite; see for instance the discussion in [5, Section 2]. It remains to show that $A_G \circ \hat{X} = 0$. There is an orthonormal basis $A_0, \dots, A_d$ of $\mathcal{A}_G$ and an index $k \in [d]$ such that A_0 is a multiple of I, the support of each matrix in $A_1, \dots, A_k$ is contained in the support of A_G, and the support of each matrix in $A_{k+1}, \dots, A_d$ is contained in the support of $A_{\overline{G}}$. Since $A_G \circ X' = 0$, we have $A_i \circ X' = 0$ and thus $\langle A_i, X' \rangle = 0$ for each $i \in [k]$. Thus, some terms in the projection $\hat{X} = \sum_{i=0}^{d} \langle A_i, X' \rangle\, A_i$ can be dropped: $\hat{X} = \langle A_0, X' \rangle\, A_0 + \sum_{i=k+1}^{d} \langle A_i, X' \rangle\, A_i$. This shows that $A_G \circ \hat{X} = 0$. □

2.1. The theta of a cartesian product

We start this section addressing a subtle point. In its original definition, ϑ_k is presented only for $k \in \mathbb{Z}_+$. However, definition (5) is immediately extended for $k \in \mathbb{R}_+$, and we shall adopt this direction in this paper. In fact, Sinjorgo and Sotirov [25, Section 3] prove that $\vartheta_k(G) \leq n$ for all k (integer), and that $\vartheta_k(G) \leq \vartheta_{k+1}(G)$, with equality if and only if $\vartheta_k(G) = n$. Carli Silva et al. [6, Lemma 1] showed the same holds for real k.

Gvozdenović and Laurent [12] derive lower bounds on χ from ϑ by computing $\vartheta(G \,\square\, K_k)$ for each integer $k \geq 1$. While doing so, they obtain $\vartheta(G \,\square\, K_k)$ as the optimal value of the following SDP (see [12, Theorem 2.7]):

$$\begin{aligned} \vartheta(G \,\square\, K_k) = \text{Maximize} \quad & k\,\langle J, X \rangle + k(k-1)\,\langle J, Y \rangle\,, \\ \text{subject to} \quad & X,\, Y \in \mathbb{S}^V, \\ & X \succcurlyeq Y, \\ & X + (k-1)Y \succcurlyeq 0, \\ & A_G \circ X = 0, \\ & I \circ Y = 0, \\ & k\,\langle I, X \rangle = 1. \end{aligned} \tag{9}$$

This SDP is well defined for any real $k \in [1, n]$. We will adopt the notation $\vartheta(G \,\square\, K_k)$ as in (9) for any real $k \in [1, n]$, even though there is no corresponding complete graph K_k if k is not an integer.

Sinjorgo and Sotirov [25], while applying the approach by Gvozdenović and Laurent [12] to lower bound the chromatic number, compared the parameter $\vartheta(G \,\square\, K_k)$ with the Narasimhan–Manber parameter $\vartheta_k(G)$. Here we extend a result they showed for graphs which are vertex- and edge-transitive.

Theorem 5. Let G be a graph on n vertices whose adjacency matrix belongs to a homogeneous partially coherent algebra, and let $k \geq 1$ be a real number. Then

$$\vartheta(G \,\square\, K_k) = \min\{k\vartheta(G), n\} = \vartheta_k(G). \tag{10}$$

Proof. Throughout this proof we denote $\vartheta \coloneqq \vartheta(G)$ and $\overline{\vartheta} \coloneqq \vartheta(\overline{G})$.

We first prove that

$$\vartheta_k(G) = \min\{k\vartheta(G), n\}, \tag{11}$$

starting with '$\leq$'. If X is feasible in (5), then $X \preccurlyeq I$ so its objective value is $\langle J, X\rangle \leq \langle J, I\rangle = n$. Moreover, $\frac{1}{k}X$ is feasible in (6) with objective value $\frac{1}{k}\langle J, X\rangle$. This shows that $\vartheta \geq \frac{1}{k}\vartheta_k(G)$. Now we prove '$\geq$'. By Lemma 4, there is an optimal solution $\hat{X}$ for (6) with constant diagonal. We will show that $\hat{Z} \coloneqq \min\{k, \overline{\vartheta}\}\, \hat{X}$ is feasible in (5) having as its objective value the RHS of (11). From Theorem 3 we have $n = \vartheta\overline{\vartheta}$, so (7) shows that $\hat{X} \preccurlyeq (1/\overline{\vartheta})I$. Hence, $\hat{Z} \preccurlyeq \overline{\vartheta}\hat{X} \preccurlyeq I$, so $\hat{Z}$ is feasible in (5) with trace constraint relaxed to $\leq k$. Its objective value is $\langle J, \hat{Z}\rangle = \min\{k, \overline{\vartheta}\}\langle J, \hat{X}\rangle = \min\{k, \overline{\vartheta}\}\,\vartheta = \min\{k\vartheta, n\}$. This concludes the proof of (11).

Now we prove

$$\vartheta(G \,\square\, K_k) = \min\{k\vartheta(G), n\}, \tag{12}$$

again starting with '$\leq$'. Let $(\hat{X}, \hat{Y})$ be feasible in (9). Note that $\hat{X} \succcurlyeq 0$ follows from $\hat{X} + (k-1)\hat{Y} \succcurlyeq 0$ and $\hat{X} - \hat{Y} \succcurlyeq 0$ via $\hat{X} = \frac{1}{k}(\hat{X} + (k-1)\hat{Y} + (k-1)(\hat{X} - \hat{Y}))$. Thus $k\hat{X}$ is feasible in (6), so the objective value of $(\hat{X}, \hat{Y})$ is

$$k\langle J, \hat{X}\rangle + k(k-1)\langle J, \hat{Y}\rangle \leq k\langle J, \hat{X}\rangle + k(k-1)\langle J, \hat{X}\rangle = k\langle J, k\hat{X}\rangle \leq k\vartheta.$$

Expanding the inner product $\langle \hat{X} + (k-1)\hat{Y}, k(nI - J)\rangle \geq 0$ while using $\langle I, \hat{Y}\rangle = 0$ and $k\langle I, \hat{X}\rangle = 1$ shows that the objective value of $(\hat{X}, \hat{Y})$ is $\leq n$.

It remains to prove '$\geq$' in (12). Let $\hat{X}$ be an optimal solution for ϑ in (6) with constant diagonal, which exists by Lemma 4. If $k = 1$, then (9) reduces to the usual SDP (6) for $\vartheta(G)$, so assume $k > 1$. Let $\lambda \coloneqq \max\{k, \overline{\vartheta}\} > 1$. Define

$$X_0 \coloneqq \frac{1}{k}\hat{X}, \qquad Y_0 \coloneqq \frac{1}{k(\lambda - 1)}\left(\frac{1}{n}J - \hat{X}\right).$$

We claim that (X_0, Y_0) is feasible in (9), with objective value $\min\{k\vartheta, n\}$. The constraints $A_G \circ X_0 = 0$, $I \circ Y_0 = 0$, and $k\langle I, X_0\rangle = 1$ follow immediately from the definition of $\hat{X}$. It remains to check the two semidefinite constraints. We have

$$X_0 - Y_0 = \frac{1}{k(\lambda - 1)}\left(\lambda\hat{X} - \frac{1}{n}J\right).$$

In the orthogonal complement of $\mathbb{1}$ this is clearly positive semidefinite, and (8) implies that the vector $\mathbb{1}$ is an eigenvector of $\lambda\hat{X} - \frac{1}{n}J$ with eigenvalue

$$\frac{\lambda\vartheta}{n} - 1,$$

which is nonnegative because

$$\lambda \geq \overline{\vartheta} = \frac{n}{\vartheta}.$$

Thus $X_0 - Y_0 \succcurlyeq 0$. Similarly,

$$X_0 + (k-1)Y_0 = \frac{1}{k(\lambda - 1)}\left((\lambda - k)\hat{X} + \frac{k-1}{n}J\right).$$

Since $\lambda \geq k$, and since both $\hat{X}$ and J are positive semidefinite, we get

$$X_0 + (k-1)Y_0 \succcurlyeq 0.$$

We now compute its objective value:

$$k\langle J, X_0\rangle + k(k-1)\langle J, Y_0\rangle = \langle J, \hat{X}\rangle + \frac{k-1}{\lambda-1}\Big\langle J, \frac{1}{n}J - \hat{X}\Big\rangle = \vartheta + \frac{k-1}{\lambda-1}(n-\vartheta).$$

If $\lambda = \overline{\vartheta} = n/\vartheta$, then

$$\vartheta + \frac{k-1}{\overline{\vartheta}-1}(n-\vartheta) = \vartheta + (k-1)\vartheta = k\vartheta.$$

If $\lambda = k$, then

$$\vartheta + \frac{k-1}{k-1}(n-\vartheta) = n.$$

Since $\lambda = \max\{k, \overline{\vartheta}\}$, these two cases give exactly

$$k\langle J, X_0\rangle + k(k-1)\langle J, Y_0\rangle = \min\{k\vartheta, n\}.$$

Therefore

$$\vartheta(G \,\square\, K_k) \geq \min\{k\vartheta, n\}.$$

Together with the upper bound already proved, this gives

$$\vartheta(G \,\square\, K_k) = \min\{k\vartheta(G), n\}. \qquad \square$$

2.2. The μ parameter

Let G be a graph on n vertices. Define

$$\mu(G) \coloneqq \min\{\, k \in [1,n] : \vartheta_k(G) = n\}. \tag{13}$$

Note that [25, Equation (39)] defines

$$\Psi_{\vartheta_k}(G) = \min\{\, k \in \mathbb{N} : \vartheta_k(G) = n\};$$

the parameter $\mu(G)$ is the continuous extension of this, with $\lceil \mu(G)\rceil = \Psi_{\vartheta_k}(G)$.

From $\alpha_k(G) \leq \vartheta_k(G)$, it follows that

$$\mu(G) \leq \chi(G).$$

The well known fact that $\vartheta(\overline{G}) \leq \chi(G)$ motivates the question:

How do $\mu(G)$ and $\vartheta(\overline{G})$ compare?

In order to address this question, we first introduce a formulation of $\mu(G)$ as an SDP, which follows immediately from (5) and relies crucially on allowing a continuous range for k in $[1,n]$:

$$\mu(G) = \min\Big\{\, \langle I, X\rangle : 0 \preccurlyeq X \preccurlyeq I,\ A_G \circ X = 0,\ \langle J, X\rangle = n\Big\}. \tag{14}$$

Theorem 6. Let $G = (V,E)$ be a graph on n vertices. If there exists an optimal solution for $\vartheta(G)$ in (6) with constant diagonal, then

$$\mu(G) = \frac{n}{\vartheta(G)}.$$

Proof. Let $\hat{X}$ be an optimal solution for $\vartheta \coloneqq \vartheta(G)$ in (6) with constant diagonal. From Lemma 4 it follows that $(n/\vartheta)\hat{X}$ is feasible for (14) with objective value n/ϑ, thus

$$\mu(G) \leq \frac{n}{\vartheta}.$$

On the other hand, it is known (and easy to see) that

$$\frac{1}{\vartheta} = \min\Big\{\, \langle I, Y\rangle : \langle J, Y\rangle = 1,\ A_G \circ Y = 0,\ Y \succcurlyeq 0\Big\}. \tag{15}$$

Thus, if X is feasible in (14), then $\frac{1}{n}X$ is feasible in (15) with objective value $\frac{1}{n}\langle I, X\rangle$. Hence

$$\frac{1}{\vartheta} \leq \frac{\mu(G)}{n}. \qquad \square$$

The second part of the proof above shows that, for any graph G on n vertices,

$$\vartheta(G)\mu(G) \geq n. \tag{16}$$

Corollary 7. If G is a graph whose adjacency matrix belongs to a homogeneous partially coherent algebra, then

$$\mu(G) = \vartheta(\overline{G}).$$

Proof. Immediate from the result above, with Theorem 3 and Lemma 4. $\square$

Computations by Pedro Cipriano first suggested that there are graphs for which $\mu(G) \neq \vartheta(\overline{G})$. Indeed, an explicit example is obtained by taking G to be the disjoint union $K_2 \cup K_1$, so that $\overline{G} = K_{1,2}$. Thus $\vartheta(\overline{G}) = 2$. However, since

$$X := \frac{1}{3}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is feasible in (14) with objective value 5/3, we obtain $\mu(G) \leq 5/3 < 2 = \vartheta(\overline{G})$.

We were not able to find an example in which $\mu(G) > \vartheta(\overline{G})$, thus we are willing to conjecture that for all graphs, $\mu(G) \leq \vartheta(\overline{G})$.

Using (16) and the fact that $\vartheta(G)\vartheta(\overline{G}) \geq n$ for every graph G on n vertices, if the conjecture holds, we would get $\frac{n}{\vartheta(G)} \leq \mu(G) \leq \vartheta(\overline{G})$.

3. The φ_k parameter and orthonormal representations

Among the many equivalent definitions of $\vartheta(G)$, we draw attention to the following (see [17, Theorem 6])

$$\vartheta(G) = \max\left\{ \lambda_1(Y) : Y \in \mathbb{S}^V_+,\ \operatorname{diag} Y = \mathbb{1},\ A_G \circ Y = 0 \right\}.$$

Motivated by this, and in the hopes of finding an equivalent characterization of ϑ_k, we introduce the parameter below for any positive integer k:

$$\varphi_k(G) := \max\left\{ \sum_{i=1}^{k} \lambda_i(Y) : Y \in \mathbb{S}^V_+,\ \operatorname{diag} Y = \mathbb{1},\ A_G \circ Y = 0 \right\}. \tag{17}$$

By Ky Fan's characterization (see (4)), we may extend this to nonnegative real k by

$$\varphi_k(G) := \max\left\{ \langle M, Y\rangle : M, Y \in \mathbb{S}^V_+,\ \operatorname{diag} Y = \mathbb{1},\ A_G \circ Y = 0,\ \langle I, M\rangle = k,\ M \preccurlyeq I \right\}. \tag{18}$$

Theorem 8. For any graph G and positive integer k,

$$\alpha_k(G) \leq \varphi_k(G) \leq \vartheta_k(G).$$

The second inequality also holds for k a nonnegative real.

Proof. To see the first inequality, consider a maximum induced subgraph of G that is k-colorable, and define the matrix $Y \in \mathbb{S}^V$ by introducing square blocks of 1s corresponding to each color class, padding the remaining diagonal entries also with 1s, and 0s otherwise. It is immediate to see that Y is feasible for (17) with objective value α_k.

For the second inequality, let k be a nonnegative real, and let $(\hat{Y}, \hat{M})$ be optimal in (18). Let $\hat{X} := \hat{Y} \circ \hat{M}$. We will prove that $\hat{X}$ is feasible for (5) with objective value $\varphi_k(G)$. We have $\langle I, \hat{X}\rangle = \langle I, \hat{M}\rangle = k$ using that $\operatorname{diag} \hat{Y} = \mathbb{1}$. Since $A_G \circ \hat{Y} = 0$, we also have $A_G \circ \hat{X} = 0$. That $\hat{X}$

is positive semidefinite follows from a theorem of Schur (see [24]), as both $\hat{Y}$ and $\hat{M}$ are positive semidefinite. Finally, $\operatorname{diag}\hat{Y} = \mathbb{1}$ implies that

$$I - \hat{X} = I - (\hat{Y} \circ \hat{M}) = \hat{Y} \circ (I - \hat{M});$$

since both of these factors are positive semidefinite, it follows that $\hat{X} \preccurlyeq I$. Thus, $\hat{X}$ is feasible, and its objective value is

$$\left\langle J, \hat{X} \right\rangle = \left\langle J, \hat{Y} \circ \hat{M} \right\rangle = \left\langle \hat{Y}, \hat{M} \right\rangle = \varphi_k(G). \qquad \square$$

We now relate φ_k to orthonormal representations. An *orthonormal representation* of G is a map $u \colon V(G) \to \mathbb{R}^m$ (for some m) such that $\|u_i\| = 1$ for all $i \in V(G)$ and $\langle u_i, u_j \rangle = 0$ whenever $ij \in E(\overline{G})$.

We provide an equivalent definition for φ_k, following the known correspondence between $\vartheta = \varphi_1$ and orthonormal representations.

Theorem 9. Let $G = (V, E)$ be a graph on n vertices. Then

$$\varphi_k(G) = \max\left\{ \sum_{i \in V} u_i^\mathsf{T} M u_i : \begin{array}{l} u \text{ is an } n\text{-dim orthonormal representation of } \overline{G}, \\ \langle I, M \rangle = k,\ 0 \preccurlyeq M \preccurlyeq I. \end{array} \right\}.$$

Proof. Let $(\hat{M}, \hat{Y})$ be a feasible solution for (18). Write $\hat{Y} = U^\mathsf{T} U$ for some matrix U, and set u_i to be ith column of U for each $i \in V$. Thus, u is an orthonormal representation of $\overline{G}$. Let $\bar{M}$ be an optimal solution for $\Lambda_k(UU^\mathsf{T})$ in (4). Since UU^T and $U^\mathsf{T}U$ have the same nonzero eigenvalues,

$$\left\langle \hat{M}, \hat{Y} \right\rangle \leq \Lambda_k(U^\mathsf{T} U) = \Lambda_k(UU^\mathsf{T}) = \left\langle \bar{M}, UU^\mathsf{T} \right\rangle = \operatorname{Tr}(U^\mathsf{T} \bar{M} U) = \sum_{i \in V} u_i^\mathsf{T} \bar{M} u_i.$$

This proves '$\leq$'. The proof of the reverse inequality is symmetric. $\square$

Note that if k is a positive integer, the theorem above immediately implies that

$$\varphi_k(G) = \max\left\{ \sum_{i \in V} u_i^\mathsf{T} P u_i : \begin{array}{l} u \text{ is an } n\text{-dim orthonormal representation of } \overline{G}, \\ P \text{ is a rank-}k \text{ orthogonal projector.} \end{array} \right\}.$$

A key parameter in the study of orthonormal representations of graphs is the minimum dimension in which such a representation exists. Let $\xi(G)$ denote such parameter. It is known [17] that

$$\vartheta(G) \leq \xi(G) \leq \chi(\overline{G}).$$

Theorem 10. Let G be a graph on n vertices. Then

$$\xi(\overline{G}) = \min\{k \in \mathbb{Z}_+ : \varphi_k(G) = n\}. \tag{19}$$

Proof. We use the fact that the rank-d feasible matrices in (17) are precisely the Gram matrices of orthonormal representations of $\overline{G}$ that span d dimensions; see, e.g., [16, Lemma 2.2]. Hence, $\xi(\overline{G})$ is the minimum rank in this set.

Let Y be any such matrix, and write it as $Y = U^\mathsf{T} U$ for some $[d] \times V$ matrix U, so that the columns of U form an orthonormal representation of $\overline{G}$. Since $\operatorname{Tr} U^\mathsf{T} U = n$, for any $k \geq 0$ we have

$$\sum_{i=1}^{k} \lambda_i(U^\mathsf{T} U) = n \iff k \geq d. \qquad \square$$

Corollary 11. For every graph G,

$$\mu(G) \leq \xi(\overline{G}). \tag{20}$$

Proof. Let G have n vertices. If $k \in \mathbb{Z}_+$ is such that $\varphi_k(G) = n$, then $\vartheta_k(G) = n$ follows from $\varphi_k(G) \leq \vartheta_k(G)$ in Theorem 8. Thus, every feasible k in the RHS of (19) is feasible in (3). The result now follows from Theorem 10. $\square$

In the RHS of (19), one might consider allowing non-integer k to be feasible, analogously to what we do in (13) for ϑ_k. However, doing so does not change the minimum value (see [23, Theorem 5.12]).

4. $\vartheta_k(G)$ does not distinguish fractional chromatic number

We recall the definition of the fractional chromatic number $\chi_f(G)$ of a graph $G = (V, E)$. Let $\mathcal{S}(G)$ be the set of nonempty cocliques of G. Then $\chi_f(G)$ is the optimal value of the following linear program (LP):

$$\chi_f(G) \coloneqq \min \left\{ \sum_{S \in \mathcal{S}(G)} y_S : y \in \mathbb{R}_+^{\mathcal{S}(G)}, \sum_{S \in \mathcal{S}(G)} y_S \mathbb{1}_S = \mathbb{1} \right\}. \tag{21}$$

Here, $\mathbb{1}_S \in \{0,1\}^V$ denotes the incidence vector of $S \subseteq V$. It is well known that $\chi_f(G) \leq \chi(G)$.

For each real $k \geq 1$, let $\alpha'_k(G)$ be the largest number of vertices in an induced subgraph of G whose fractional chromatic number is $\leq k$. Note that $\alpha'_k(G) \geq \alpha_k(G)$. The motivating question of this section is whether $\vartheta_k(G)$ can distinguish k-colorability from having fractional chromatic number at most k, that is, whether there exists a graph G such that $\alpha'_k(G) > \vartheta_k(G)$. The following result shows that this is not the case.

Theorem 12. For every graph G and every real $k \geq 1$,

$$\alpha'_k(G) \leq \vartheta_k(G).$$

Proof. Let H be an induced subgraph of G with $|V(H)| = \alpha'_k(G)$ and $\chi_f(H) \leq k$, and let $y \in \mathbb{R}_+^{\mathcal{S}(H)}$ be an optimal solution for the LP (21) defining $\chi_f(H)$. Define $\hat{X} \in \mathbb{S}^V$ by

$$\hat{X} \coloneqq \sum_{S \in \mathcal{S}(H)} y_S \frac{\mathbb{1}_S \mathbb{1}_S^\top}{|S|}.$$

It follows immediately that $\hat{X} \succcurlyeq 0$, $A_G \circ \hat{X} = 0$, and $\langle I, \hat{X} \rangle = \sum_{S \in \mathcal{S}(H)} y_S = \chi_f(H) \leq k$. From the LP formulation,

$$\hat{X} \ \mathbb{1}_{V(H)} = \sum_{S \in \mathcal{S}(H)} y_S \mathbb{1}_S = \mathbb{1}_{V(H)},$$

and because $\hat{X}$ is a nonnegative matrix, the Perron-Frobenius theorem (see for instance [3, Section 2.2]) implies that the largest eigenvalue of $\hat{X}$ is 1, hence $\hat{X} \preccurlyeq I$. Finally,

$$\left\langle J, \hat{X} \right\rangle = \sum_{S \in \mathcal{S}(H)} y_S |S| = (\mathbb{1}_{V(H)})^\top \left(\sum_{S \in \mathcal{S}(H)} y_S \mathbb{1}_S \right) = |V(H)| = \alpha'_k(G).$$

Hence if we let $\ell \coloneqq \chi_f(H)$, then $\hat{X}$ is feasible for the SDP (5) defining $\vartheta_\ell(G)$ with objective value $\alpha'_k(G)$. Combining with monotonicity of $\vartheta_k(G)$ with respect to k, we obtain $\alpha'_k(G) \leq \vartheta_\ell(G) \leq \vartheta_k(G)$ since $\ell \leq k$. □

5. The Weighted $\vartheta_k(G, \cdot)$ Function

The Lovász theta graph parameter was extended to a real-valued function over vectors of nonnegative vertex weights in [11]. This function was shown to be a positive definite monotone gauge, and this led to a rich theory connecting polyhedral combinatorics and convex optimization (see [2] for a self-contained review). Alizadeh [1] introduced a vertex-weighted version of ϑ_k. In this section we briefly review known results and basic properties, connect them to a newly introduced vertex weighted version of φ_k, show a connection between these and χ_f, but more notably we show that $w \mapsto \vartheta_k(G, w)$ is not a gauge. While a negative result is perhaps not exciting, we expect it to clarify the geometric limitations of $\vartheta_k(G,\cdot)$ as a relaxation, and to guide the development of alternative formulations that retain desirable convex–analytic properties, such as gauge or norm structures.

Throughout this section, $w \in \mathbb{R}^V_+$ denotes a nonnegative vector of vertex weights. Define $\sqrt{w} \in \mathbb{R}^V_+$ as the vector obtained by taking the entry-wise square root, and

$$W \coloneqq \sqrt{w}\sqrt{w}^{\mathsf{T}}.$$

We refer to the Master's dissertation [23] of the third author for a detailed treatment of the results in this section.

5.1. **A sandwich theorem and connection to χ_f**

Alizadeh [1] defined

$$\vartheta_k(G,w) \coloneqq \max\{\, \langle W, X\rangle \,:\, \langle I, X\rangle = k,\, A_G \circ X = 0,\, 0 \preccurlyeq X \preccurlyeq I\}.$$

Here we introduce three definitions in order to obtain a vertex-weighted sandwich theorem generalizing $\alpha_k(G) \le \varphi_k(G) \le \vartheta_k(G) \le \chi_k(\overline{G})$.

Define

$$\varphi_k(G,w) \coloneqq \max\left\{\, \langle M, Y\rangle : M, Y \in \mathbb{S}^V_+,\, \operatorname{diag} Y = w,\, A_G \circ Y = 0,\, \langle I, M\rangle = k,\, M \preccurlyeq I\right\}. \tag{22}$$

On the combinatorial side, for each integer k, set

$$\alpha_k(G,w) \coloneqq \max\left\{\sum_{i\in U} w_i : U \subseteq V(G),\ G[U] \text{ is } k\text{-colorable}\right\}.$$

Recall that $\mathcal{S}(G)$ denotes the set of nonempty cocliques of G. Define, for each $w \in \mathbb{Z}^V_+$,

$$\chi_k(G,w) \coloneqq \min\left\{\sum_{S\in\mathcal{S}(G)} y_S : y \in \mathbb{Z}^{\mathcal{S}(G)}_+,\ \sum_{S\in\mathcal{S}(G)} y_S \mathbb{1}_S = k\,w\right\}.$$

The best argument for why these are the correct generalizations is the theorem below and its proof, which is a natural generalization of standard arguments both in the case $k = 1$ for general weights, and in the case $k > 1$ for $w = \mathbb{1}$. Because the proof does not bring any new relevant insight and because it is slightly technical in dealing with corner cases (vertices for which $w_i = 0$, for instance), we prefer to omit it here, and refer the reader to [23, Theorem 5.2].

Theorem 13. Let $G = (V, E)$ be a graph and let k be a nonnegative integer. Then for every $w \in \mathbb{R}^V_+$,

$$\alpha_k(G,w) \le \varphi_k(G,w) \le \vartheta_k(G,w).$$

The second inequality also holds for nonnegative real k. If $w \in \mathbb{Z}^V_+$ and $k \in \mathbb{Z}_+$, then

$$\vartheta_k(G,w) \le \chi_k(\overline{G},w).$$

We introduce a vertex-weighted version of the parameter μ studied earlier:

$$\mu(G,w) \coloneqq \min\{k \in \mathbb{R}_+ : \vartheta_k(G,w) = \langle w, \mathbb{1}\rangle\}. \tag{23}$$

In [6, Theorem 4] (see also [23, Proposition 4.7]), the following result is proved.

Theorem 14. Let $G = (V, E)$ be a graph. For all $w \in \mathbb{R}^V_+$,

$$\mu(G,w) \le \chi_f(G).$$

Since $\mu(G) = \mu(G, \mathbb{1})$, the above result can be used to separate the inequality (20). Namely, by taking G to be the 5-cycle, we get $\mu(G) \le \chi_f(G) < \xi(\overline{G})$. We refer the reader to [23, Proposition 5.13] for further details.

5.2. Parameter $\vartheta_k(G,\cdot)$ is not a gauge

A function $\kappa\colon \mathbb{R}^V_+ \to \mathbb{R}$ is a positive definite gauge if κ is nonnegative, zero-valued only at the origin, positively homogeneous, and subadditive. That is, $\kappa(x) \geq 0$ for each $x \in \mathbb{R}^V_+$, with equality if and only if $x = 0$, $\kappa(\lambda x) = \lambda\kappa(x)$ for each $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^V_+$, and $\kappa(x+y) \leq \kappa(x) + \kappa(y)$ for each $x, y \in \mathbb{R}^V_+$. We call κ monotone if $\kappa(x) \leq \kappa(y)$ whenever $x \leq y$ (entry-wise). Positive definite monotone gauges are essentially the same as norms restricted to the nonnegative orthant, and they are a fundamental tool in combinatorial optimization. See [2] for a self-contained review on gauges and their connection to combinatorial optimization.

For $k = 1$, the map $w \mapsto \vartheta_1(G, w)$ is a positive definite monotone gauge. Its unit convex corner, that is, $\{\, w \in \mathbb{R}^V_+ : \vartheta_1(G,w) \leq 1\}$, is the theta body of the complement graph, and this has been extensively studied since its introduction in [11].

For $k > 1$, this fails in general.

Proposition 15. There are a graph G, an integer $k > 1$, and weights $w, z \in \mathbb{Z}^V_+$ such that

$$\vartheta_k(G, w+z) > \vartheta_k(G,w) + \vartheta_k(G,z).$$

Proof. Take G to be the disjoint union $K_3 \cup K_1$, $k \coloneqq 2$, and $w \coloneqq \mathbb{1}$. Define z as having weight 1 on the vertices of K_3 and weight 3 on the single vertex of K_1. From the explicit formula for $\vartheta_k(K_n \cup K_1)$ proved in [23], we have

$$\vartheta_2(G,w) = 2 + \sqrt{3}.$$

Consider

$$X \coloneqq \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 1 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is feasible for $\vartheta_2(G,z)$ and has objective value

$$\left\langle \sqrt{z}\sqrt{z}^{\mathsf{T}}, X \right\rangle = 6 = \langle z, \mathbb{1} \rangle\,,$$

so $\vartheta_2(G,z) = 6$. Now evaluate the same feasible matrix for $w+z$:

$$\vartheta_2(G, w+z) \geq \left\langle \sqrt{w+z}\sqrt{w+z}^{\mathsf{T}}, X \right\rangle = 5 + 2\sqrt{6}.$$

Hence

$$\vartheta_2(G, w+z) \geq 5 + 2\sqrt{6} > 8 + \sqrt{3} = \vartheta_2(G,w) + \vartheta_2(G,z). \qquad \square$$

As a consequence, one cannot use $\vartheta_k(G,\cdot)$ the same way $\vartheta(G,\cdot)$ is used to define an analogous version of a k-fold theta body. We discuss in the next section possible alternative approaches to define a k-fold theta body.

5.3. The parameter $\varphi_k(G,\cdot)$ is a gauge

The parameter $\varphi_k(G,\cdot)$ retains the gauge structure for $k > 1$.

Theorem 16. Let G be a graph and let $k > 0$. Then the map $w \mapsto \varphi_k(G,w)$ defined in (22) is a positive definite monotone gauge.

Proof. Let $w, z \in \mathbb{R}^V_+$ and let $\lambda \geq 0$. We will show that $\varphi_k(G,\cdot)$ is positive definite, positively homogeneous, subadditive, and monotone.

Positive definiteness: the facts that $\varphi_k(G,\cdot)$ is nonnegative and $\varphi_k(G,0) = 0$ are immediate. If $w \neq 0$, then any matrix Y with $\operatorname{diag} Y = w$ has a positive eigenvalue. By choosing M to contain a positive eigenvalue in the same eigenspace, we can ensure that $\langle M, Y\rangle > 0$, hence $\varphi_k(G,w) > 0$.

Positive homogeneity: let $(\hat{M}, \hat{Y})$ be feasible for $\varphi_k(G, w)$ in (22), attaining the optimum. Note that $(\hat{M}, \lambda\hat{Y})$ is feasible for $\varphi_k(G, \lambda w)$ with objective value $\lambda\varphi_k(G, w)$. Thus, we have $\varphi_k(G, \lambda w) \geq \lambda\varphi_k(G, w)$. On the other hand, if $(\hat{M}, \hat{Y})$ is feasible for $\varphi_k(G, \lambda w)$, with $\lambda \neq 0$, and attaining the optimum, then $(\hat{M}, \frac{1}{\lambda}\hat{Y})$ is feasible for $\varphi_k(G, w)$ and its objective value is $\frac{1}{\lambda}\varphi_k(G, \lambda w)$. Thus $\varphi_k(G, \lambda w) = \lambda\varphi_k(G, w)$.

To see subadditivity, we resort to an alternative formulation of $\varphi_k(G, w)$. Let us define the set

$$\mathtt{M}_k(G) \coloneqq \left\{ x \in \mathbb{R}^V_+ : \begin{array}{l} \exists\ M \in \mathbb{S}^V_+, M \preccurlyeq I, \langle I, M\rangle = k, \\ \exists\ u \colon V \to \mathbb{R}^V \text{ an orthonormal representation of } \overline{G} \\ \text{with } x_i = u_i^\mathsf{T} M u_i \end{array} \right\}.$$

An immediate extension of Theorem 9 shows that

$$\varphi_k(G, w) = \max\{\langle w, x\rangle : x \in \mathtt{M}_k(G)\}.$$

(See [23, Theorem 5.11] for a detailed proof of this equivalence.)

Subadditivity: follows from

$$\begin{aligned} \varphi_k(G, w + z) &= \max\{\langle w + z, x\rangle : x \in \mathtt{M}_k(G)\} \\ &= \max\{\langle w, x\rangle + \langle z, x\rangle : x \in \mathtt{M}_k(G)\} \\ &\leq \max\{\langle w, x\rangle : x \in \mathtt{M}_k(G)\} + \max\{\langle z, x\rangle : x \in \mathtt{M}_k(G)\} \\ &= \varphi_k(G, w) + \varphi_k(G, z). \end{aligned}$$

Monotonicity: if $w \leq z$, then $\langle w, x\rangle \leq \langle z, x\rangle$ for every $x \in \mathtt{M}_k(G)$. Hence $\varphi_k(G, w) = \max\{\langle w, x\rangle : x \in \mathtt{M}_k(G)\} \leq \max\{\langle z, x\rangle : x \in \mathtt{M}_k(G)\} = \varphi_k(G, z)$.

□

While the result above offers a strong perspective on the extension of the $k = 1$ theory for general k, we do not know whether the set $\mathtt{M}_k(G)$ is SDP representable for $k > 1$. In fact, it is not even clear whether $\mathtt{M}_k(G)$ is convex for $k > 1$. This is the motivation for Section 6.

5.4. A Greene–Kleitman–Lovász upper bound to $\varphi_k(G, w)$

Greene [9] and Greene–Kleitman [10] studied a colouring parameter that Lovász later used in his definition of k-perfect graphs. For a graph $G = (V, E)$, let

$$\psi_k(G) = \min_{S \subseteq V}\{k\chi(G - S) + |S|\}.$$

If G is a comparability graph, then cliques are chains and

$$\min_{\mathcal{C}} \sum_{C \in \mathcal{C}} \min\{|C|, k\} = \min_{S \subseteq V}\Big\{k\chi(\overline{G} - S) + |S|\Big\},$$

where the leftmost minimum is taken over all chain partitions $\mathcal{C}$.

We proceed to define the fractional weighted version for a vector $w \in \mathbb{R}^V_+$ and a positive integer k by

$$\psi_k^f(G, w) \coloneqq \min_{S \subseteq V}\left\{k\ \chi_f(G - S, w_{V\setminus S}) + \sum_{i \in S} w_i\right\}.$$

The parameter interpolates between colouring and vertex deletion: one may delete a set S, paying its full weight, and then pay k times the (fractional) colouring cost of the remaining graph. Lovász's class of k-perfect graphs is motivated by equality between α_k and the ordinary version of this parameter on all induced subgraphs.

Theorem 17. Let $G = (V, E)$ be a graph on n vertices, let $w \in \mathbb{R}^V_+$, and let $k \in [n]$. Then

$$\varphi_k(G, w) \leq \psi_k^f(\overline{G}, w).$$

Proof. Fix $S \subseteq V$. We split the proof in two cases. If $k > n - |S|$, then

$$\begin{aligned}\varphi_k(G,w) &\le w^{\mathsf{T}}\mathbb{1}\\ &\le (n-|S|)\ \max\{w_i : i \in V \setminus S\} + w^{\mathsf{T}}\mathbb{1}_S\\ &\le k\ \chi_f(\overline{G}-S, w_{V\setminus S}) + w^{\mathsf{T}}\mathbb{1}_S.\end{aligned}$$

Otherwise $k \le n - |S|$. Let $\widehat{Y}$ be optimum for (22), and $\widehat{Y}_{\overline{S}}$ its principal submatrix corresponding to $V \setminus S$. By eigenvalue interlacing applied to a principal submatrix, one has

$$\operatorname{Tr}\widehat{Y}_{\overline{S}} - \sum_{i=1}^{k}\lambda_i(\widehat{Y}_{\overline{S}}) \le \operatorname{Tr}\widehat{Y} - \sum_{i=1}^{k}\lambda_i(\widehat{Y}),$$

thus

$$\begin{aligned}\varphi_k(G,w) &\le \sum_{i=1}^{k}\lambda_i(\widehat{Y}_{\overline{S}}) + (\operatorname{Tr}\widehat{Y} - \operatorname{Tr}\widehat{Y}_{\overline{S}})\\ &\le \varphi_k(G-S, w_{V\setminus S}) + w^{\mathsf{T}}\mathbb{1}_S.\end{aligned}$$

Also $\varphi_k(H,u) \le k\varphi_1(H,u)$ for every graph H and $u \in \mathbb{R}_+^{V(H)}$. Since $\varphi_1(H,u) = \vartheta(H,u) \le \chi_f(\overline{H},u)$, we get

$$\varphi_k(G,w) \le k\ \chi_f(\overline{G}-S, w_{V\setminus S}) + \sum_{i\in S} w_i.$$

Taking the minimum over all $S \subseteq V$ gives the result. ☐

For $G \coloneqq \overline{K_n}$, we have $\psi_k^f(\overline{G}) = \psi_k(\overline{G}) = n$ and $\chi_k(\overline{G}) = kn$. Thus, Theorem 17 is sharper than (2) on some graphs.

6. The k-fold theta body

The failure of subadditivity for $\vartheta_k(G,\cdot)$ prevents us from using ϑ_k directly to define a convex theta body in the same way that $\vartheta(G,\cdot)$ can be used to define $\mathrm{TH}(G)$. We therefore introduce an explicitly convex substitute. For an integer $k \ge 1$, our natural proposal for a k-fold theta body is given by

$$\mathrm{TH}_k(G) \coloneqq \left\{ x \in \mathbb{R}_+^V : \begin{array}{l} x = x^{(1)} + \cdots + x^{(k)},\\ x^{(i)} \in \mathrm{TH}(G) \text{ for each } i \in [k],\\ 0 \le x \le \mathbb{1}. \end{array} \right\}. \tag{24}$$

This can be naturally extended to nonnegative real k, recalling that $\mathrm{TH}(G)$ is a convex set, by

$$\mathrm{TH}_k(G) \coloneqq k\,\mathrm{TH}(G) \cap [0,1]^V. \tag{25}$$

Note that this is a convex set. We then naturally define, for each $w \in \mathbb{R}_+^V$,

$$\theta_k(G,w) \coloneqq \max\{\, \langle w, x\rangle \ : x \in \mathrm{TH}_k(G)\} \tag{26}$$

The following proposition is immediate.

Proposition 18. The function $\theta_k(G,\cdot)$ defined in (26) is a positive definite monotone gauge.

We can also show that $\theta_k(G,w)$ is an upper bound for $\alpha_k(G,w)$:

Proposition 19. For every graph $G = (V,E)$, every positive integer k, and every $w \in \mathbb{R}_+^V$,

$$\alpha_k(G,w) \le \theta_k(G,w).$$

Proof. This follows from the fact that $\mathrm{TH}(G)$ contains the characteristic vector of every stable set of G, hence $\mathrm{TH}_k(G)$ contains the characteristic vector of every k-colorable induced subgraph of G. ☐

We now show that $\theta_k(G,w)$ can be computed as the optimal value of an SDP. We highlight that a very similar SDP was proposed by Kuryatnikova, Sotirov, and Vera in [15] with an extra constraint that $Z \geq 0$, though without the motivation of defining a k-fold theta body.

Theorem 20. Let $G = (V,E)$ be a graph and let $k > 0$. Then

$$\mathrm{TH}_k(G) = \left\{ \operatorname{diag}(Z) : \begin{array}{c} Z \in \mathbb{S}^V,\ A_G \circ Z = 0,\ I \circ Z \leq I, \\ \begin{pmatrix} k & \operatorname{diag}(Z)^\mathsf{T} \\ \operatorname{diag}(Z) & Z \end{pmatrix} \succcurlyeq 0 \end{array} \right\}. \tag{27}$$

Proof. Recall that $x \in \mathrm{TH}(G)$ if and only if there exists $Y \in \mathbb{S}^V$ such that $\operatorname{diag}(Y) = x$, $A_G \circ Y = 0$, and

$$\begin{pmatrix} 1 & x^\mathsf{T} \\ x & Y \end{pmatrix} \succcurlyeq 0.$$

Hence $x \in k\ \mathrm{TH}(G)$ if and only if $\frac{1}{k}x \in \mathrm{TH}(G)$. By the previous characterization, this holds if and only if there exists $Y \in \mathbb{S}^V$ such that $\operatorname{diag}(Y) = \frac{1}{k}x$, $A_G \circ Y = 0$, and

$$\begin{pmatrix} 1 & \frac{1}{k}x^\mathsf{T} \\ \frac{1}{k}x & Y \end{pmatrix} \succcurlyeq 0.$$

Multiplying this block matrix by k, and setting $Z \coloneqq kY$, we get the equivalent conditions $\operatorname{diag}(Z) = x$, $A_G \circ Z = 0$, and

$$\begin{pmatrix} k & x^\mathsf{T} \\ x & Z \end{pmatrix} \succcurlyeq 0.$$

Therefore

$$k\ \mathrm{TH}(G) = \left\{ \operatorname{diag}(Z) : Z \in \mathbb{S}^V,\ A_G \circ Z = 0,\ \begin{pmatrix} k & \operatorname{diag}(Z)^\mathsf{T} \\ \operatorname{diag}(Z) & Z \end{pmatrix} \succcurlyeq 0 \right\}.$$

By definition, $\mathrm{TH}_k(G) = k\ \mathrm{TH}(G) \cap [0,1]^V$. Since $x = \operatorname{diag}(Z)$, the additional condition $x \leq \mathbb{1}$ is exactly $I \circ Z \leq I$. The condition $x \geq 0$ is automatic from the positive semidefiniteness of the block matrix, because $Z \succcurlyeq 0$ and hence $Z_{ii} \geq 0$ for all $i \in V$.

Combining these observations gives the desired representation. $\square$

Our main result in this section is that $\theta_k(G,w)$ is an upper bound for $\varphi_k(G,w)$.

Theorem 21. Let $G = (V,E)$ be a graph, let $k \in \mathbb{R}_+$, and let $w \in \mathbb{R}^V_+$. Then

$$\varphi_k(G,w) \leq \theta_k(G,w).$$

Proof. We use the orthonormal representation formulation of $\varphi_k(G,w)$. Namely,

$$\varphi_k(G,w) = \max\left\{ \sum_{i \in V} w_i u_i^\mathsf{T} M u_i : \begin{array}{l} u \text{ is an orthonormal representation of } \overline{G}, \\ 0 \preccurlyeq M \preccurlyeq I,\ \langle I, M \rangle = k \end{array} \right\}.$$

Let u and M be feasible for this formulation. Define $x \in \mathbb{R}^V$ by

$$x_i \coloneqq u_i^\mathsf{T} M u_i \qquad \forall i \in V.$$

We claim that $x \in \mathrm{TH}_k(G)$.

Since $0 \preccurlyeq M \preccurlyeq I$, take a spectral decomposition

$$M = \sum_{r=1}^{m} t_r c_r c_r^\mathsf{T},$$

where $c_1, \ldots, c_m$ are orthonormal vectors, $0 \leq t_r \leq 1$, and

$$\sum_{r=1}^{m} t_r = \langle I, M \rangle = k.$$

For each r, define $x^{(r)} \in \mathbb{R}^V$ by
$$x_i^{(r)} := \langle c_r, u_i\rangle^2.$$
We first show that $x^{(r)} \in \mathrm{TH}(G)$. Indeed, consider the vectors
$$g_0 := c_r, \qquad g_i := \langle c_r, u_i\rangle u_i \quad \forall i \in V.$$
Their Gram matrix $Z^{(r)}$, indexed by $\{0\} \cup V$, satisfies
$$Z_{00}^{(r)} = 1, \qquad Z_{0i}^{(r)} = x_i^{(r)} = Z_{ii}^{(r)} \quad \forall i \in V.$$
Moreover, since u is an orthonormal representation of $\overline{G}$, we have $\langle u_i, u_j\rangle = 0$ whenever $ij \in E(G)$. Hence
$$Z_{ij}^{(r)} = \langle c_r, u_i\rangle\langle c_r, u_j\rangle\langle u_i, u_j\rangle = 0 \qquad \forall ij \in E(G).$$
Thus $x^{(r)} \in \mathrm{TH}(G)$.

Now
$$x_i = u_i^\mathsf{T} M u_i = \sum_{r=1}^m t_r \langle c_r, u_i\rangle^2 = \sum_{r=1}^m t_r x_i^{(r)}.$$
Since $\sum_r t_r = k$ and $\mathrm{TH}(G)$ is convex, it follows that
$$x \in k\ \mathrm{TH}(G).$$
Also, because $0 \preccurlyeq M \preccurlyeq I$ and $\|u_i\| = 1$, we have
$$0 \le x_i = u_i^\mathsf{T} M u_i \le u_i^\mathsf{T} u_i = 1 \qquad \forall i \in V.$$
Therefore
$$x \in k\ \mathrm{TH}(G) \cap [0,1]^V = \mathrm{TH}_k(G).$$

Finally,
$$\sum_{i\in V} w_i u_i^\mathsf{T} M u_i = \sum_{i \in V} w_i x_i = \langle w, x\rangle \le \max\{\langle w, y\rangle : y \in \mathrm{TH}_k(G)\} = \theta_k(G, w).$$
Since this holds for every feasible pair (u, M), we conclude that
$$\varphi_k(G, w) \le \theta_k(G, w).$$
□

As usual, define $\theta_k(G) := \theta_k(G, \mathbb{1})$.

Proposition 22. There exist a graph G and an integer k such that
$$\varphi_k(G) < \theta_k(G).$$

Proof. Let $G := KG(6, 2)$, the Kneser graph whose vertices are the 2-subsets of [6], with two vertices adjacent when the corresponding sets are disjoint. Then $\vartheta(G) = 5$; see [17].

Averaging an optimal point of $\mathrm{TH}(G)$ over the (vertex-transitive) automorphism group gives $\frac{1}{3}\mathbb{1} \in \mathrm{TH}(G)$. Thus $\mathbb{1} \in 3\,\mathrm{TH}(G) \cap [0,1]^V = \mathrm{TH}_3(G)$, and hence
$$\theta_3(G) = 15.$$
On the other hand, $\xi(\overline{G}) = 4$ (see [13]). From Theorem 10, it follows that
$$\varphi_3(G) < 15 = \theta_3(G).$$
□

The next result can be seen as a θ_k counterpart to Theorem 5, albeit with a stronger hypothesis. We denote $\theta := \theta_1 = \vartheta$.

Corollary 23. Let G be a vertex-transitive graph on n vertices. Let $k \ge 1$ be a real number. Then
$$\theta(G \,\square\, K_k) = \min\{k\theta(G), n\} = \theta_k(G).$$

Proof. Averaging an optimal solution attaining $\theta(G)$ (on $\mathrm{TH}(G)$) over the automorphism group of G, yields

$$\frac{\theta(G)}{n}\mathbb{1} \in \mathrm{TH}(G).$$

Since $\mathrm{TH}(G)$ is a convex corner, $t\mathbb{1} \in \mathrm{TH}_k(G)$ for $t \coloneqq \min\{k\theta(G)/n, 1\}$. Thus, $\theta_k(G) = \theta_k(G, \mathbb{1}) \geq \min\{k\theta(G), n\}$. The reverse inequality follows from $\theta_k(G, w) \leq \min\{k\theta(G, w), \langle w, \mathbb{1}\rangle\}$. Applying Theorem 5 concludes the proof. $\square$

7. Conclusion

We studied several theta-type relaxations for the maximum k-colorable induced subgraph problem. The Narasimhan–Manber parameter ϑ_k is the most direct SDP generalization of Lovász theta, but it is neither subadditive as a function of vertex weights, nor naturally related to orthonormal representations. In this paper we filled this gap in the literature by exploring parameters that satisfy these roles and we proved several of their properties, leading to connections with graph products, regularity, orthonormal representations, and convex corners.

Here is a list of questions that remain open.

Open questions

(i) Compare θ_k and ϑ_k. Is it true that

$$\theta_k(G, w) \leq \vartheta_k(G, w) \qquad \forall G, k, w \geq 0?$$

Since $w \mapsto \vartheta_k(G, w)$ is not sublinear in general, a more interesting question is obtained by letting $\vartheta_k^{\mathrm{conv}}(G, \cdot)$ be the largest monotone gauge dominated by $w \mapsto \vartheta_k(G, w)$. Does

$$\vartheta_k^{\mathrm{conv}}(G, w) \leq \theta_k(G, w)$$

hold for all $G, k, w \geq 0$? Can equality hold, perhaps under additional symmetry assumptions? Kuryatnikova, Sotirov, and Vera [15] define a matrix-lifting relaxation θ_k^3, which is essentially the Schrijver-type version of the body $\mathrm{TH}_k(G)$, obtained by adding entrywise nonnegativity to the SDP with the RHS of (27) as its feasible region. They conjecture that $\theta_k^3(G) \leq \vartheta'_k(G)$, which is the Schrijver-type version of our question above.

(ii) Sinjorgo and Sotirov conjectured that

$$\vartheta(G \,\square\, K_k) \leq \vartheta_k(G),$$

with equality whenever $\vartheta_k(G) = k\vartheta(G)$. We showed this for a broad homogeneous class of graphs. Does the conjecture hold for all graphs and all real $k \in [1, n]$?

(iii) Is

$$\mu(G) \leq \vartheta(\overline{G})$$

for every graph G? Numerical evidence suggests that equality can fail, but no example is known to us with $\mu(G) > \vartheta(\overline{G})$.

(iv) Is the orthonormal representation set $\mathtt{M}_k(G)$ convex? If not, can one characterize its closed convex lower hull, i.e., the convex corner whose support function is $\varphi_k(G, \cdot)$?

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Statement of AI Use

Generative AI (ChatGPT 5.5) was used for organizing and reviewing the text. It was also used to find the example in Proposition 22. The authors are fully responsible for the entire content of this paper.

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