

Generalizations of Total Dual Integrality*

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Abstract

We design new tools to study variants of Total Dual Integrality. As an application, we obtain a geometric characterization of Total Dual Integrality for the case where the associated polyhedron is non-degenerate. We also give sufficient conditions for a system to be Totally Dual Dyadic, and prove new special cases of Seymour’s Dyadic conjecture on ideal clutters.

Keywords: Integrality of polyhedra, dual integrality, TDI systems, dyadic rationals, dense subsets, ideal clutters.

1 Introduction

Consider a system of linear inequalities $Mx \leq b$ where M, b are integral and M has m rows and n columns. For a vector $w \in \mathbb{R}^n$ define the following primal-dual pair of linear programs,

$$\max\{w^\top x : Mx \leq b\}, \quad (P : M, b, w)$$

$$\min\{b^\top y : M^\top y = w, y \geq \mathbf{0}\}. \quad (D : M, b, w)$$

Recall that from Strong Duality, both or none of these linear programs have an optimal solution. We say that w is *admissible* in the former case. Let \mathbb{L} denote a subset of the reals \mathbb{R} . We say that $Mx \leq b$ is *totally dual in \mathbb{L}* (abbreviated as TD in \mathbb{L}) if for every $w \in \mathbb{L}^n$ for which w is admissible, $(D : M, b, w)$ has an optimal solution in \mathbb{L}^m . When $\mathbb{L} = \mathbb{Z}$ this corresponds to totally dual integral (TDI) systems. We say that $\mathbb{L} \subseteq \mathbb{R}$ is *heavy* if it is a dense subset of \mathbb{R} that contains all integers and $(\mathbb{L}, +)$ forms a subgroup of the additive group of real numbers. Our key result, Theorem 2.2, yields a characterization of systems that are totally dual in \mathbb{L} when \mathbb{L} is heavy. To motivate this result, we present in the

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introduction applications, and highlights their relevance to the study of TDI systems and to a long standing conjecture of Paul Seymour on ideal clutters.

1.1 Totally dual in \mathbb{L} systems

Consider a prime p . A rational number is p -adic if it is of the form $\frac{r}{s}$ where $r, s \in \mathbb{Z}$ and s is an integer power of p [2]. A number is *dyadic* if it is 2-adic. Let \mathcal{S} be a (possibly infinite) set of primes. We denote by $\mathbb{L}(\mathcal{S})$ the set of all rationals of the form $\frac{r}{s}$ where $r, s \in \mathbb{Z}$ and s is a product of integer powers of primes in \mathcal{S} . Observe that $\mathbb{L}(\{p\})$ denotes the p -adic rationals. To keep the notation light, we write \mathbb{L}_p for $\mathbb{L}(\{p\})$. For any set of primes \mathcal{S} , $\mathbb{L}(\mathcal{S})$ is a heavy set (this follows from [3, Lemma 2.1]).

For a system $Mx \leq b$ to be TD in \mathbb{L} we consider the dual $(D: M, b, w)$ for all choices of $w \in \mathbb{L}^n$. However, for the aforementioned heavy sets, it suffices to consider the choices $w \in \mathbb{Z}^n$. Namely,

Remark 1.1. *Let $\mathbb{L} := \mathbb{L}(\mathcal{S})$ where \mathcal{S} is a set of primes. Then $Mx \leq b$ is TD in \mathbb{L} if and only if for every admissible $w \in \mathbb{Z}^n$, $(D: M, b, w)$ has an optimal solution in \mathbb{L}^n .*

Proof. Necessity is clear, let us prove sufficiency. Consider an admissible $w \in \mathbb{L}^n$. For some μ that is a product of integer powers of primes in \mathcal{S} we have $\mu w \in \mathbb{Z}^n$. There exists an optimal solution $\bar{y} \in \mathbb{L}^n$ for $(D: M, b, \mu w)$. However, then $\frac{1}{\mu}\bar{y}$ is a solution for $(D: M, b, w)$ that is in \mathbb{L}^n . \square

For any prime p , we can find a system $Mx \leq b$ that is TD in \mathbb{L}_p but not TD in \mathbb{L}_q for any prime $q \neq p$. Namely, one can pick $Mx \leq b$ to consist of a unique constraint, $px \leq 1$. However, if we require a system to be TD in \mathbb{L}_p and TD in $\mathbb{L}_{p'}$ for distinct primes p and p' , then it is totally dual in \mathbb{L}_q for every prime q [2, Theorem 1.4]. The following stronger statement (the aforementioned case corresponds to $\mathcal{S}_1 = \{p\}, \mathcal{S}_2 = \{p'\}$ and $k = 2$) holds in the full dimensional case,

Theorem 1.2. *Let $Mx \leq b$ be a system where $\{x : Mx \leq b\}$ is a full-dimensional polyhedron. For $i = 1, \dots, k$, let \mathcal{S}_i be a set of primes and suppose that $Mx \leq b$ is TD in $\mathbb{L}(\mathcal{S}_i)$. If $\bigcap_{i \in [k]} \mathcal{S}_i = \emptyset$ then $Mx \leq b$ is TD in \mathbb{L}_q for every prime q .*

This will be an immediate consequence of Theorem 2.2.

Consider an integral matrix M and an integral vector b . A necessary condition for $Mx \leq b$ to be TDI is that the polyhedron $Q = \{x \in \mathbb{R}^n : Mx \leq b\}$ be integral, i.e., that every minimal proper face of Q contains an integral vector [10], see also [20, Corollary 22.1a]. However, this is not a sufficient condition [20, Equation (3) in Chapter 22]. Let us define a stronger necessary condition for a system to be TDI. We say that $Mx \leq b$ is *near-TDI* if for every prime p , $Mx \leq b$ is TD in \mathbb{L}_p . Since $\mathbb{Z} \subset \mathbb{L}_p$ for every prime p , it then follows from Remark 1.1 that if a system is TDI, it is near-TDI.

Furthermore, we have the following result [3, Theorem 1.5],

Proposition 1.3. Consider $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. If $Mx \leq b$ is TD in \mathbb{L}_p and TD in \mathbb{L}_q for distinct primes p, q then the polyhedron $\{x : Mx \leq b\}$ is integral.

Theorem 2.2 characterizes when a system $Mx \leq b$ is TD in \mathbb{L}_p for some prime p . The proof leverages the density of \mathbb{L}_p . This suggests, unexpectedly, that density arguments may sometimes be very useful in certifying integrality of polyhedra via Proposition 1.3.

It follows from Proposition 1.3 that if $Mx \leq b$ is near-TDI then $\{x : Mx \leq b\}$ is integral. Therefore, for M, b integer, if $Mx \leq b$ is TDI then it is near-TDI, and if $Mx \leq b$ is near-TDI, then $\{x : Mx \leq b\}$ is integral.

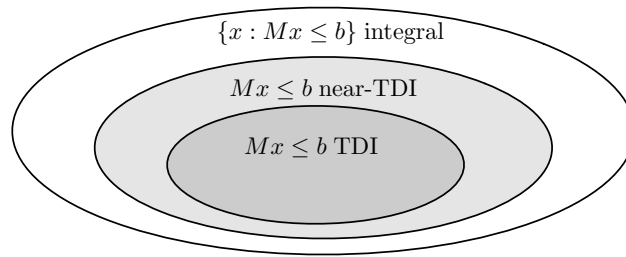


Figure 1: Hierarchy of linear systems of inequalities.

A system $Mx \leq b$ is *non-degenerate* if for every minimal proper face F of $\{x : Mx \leq b\}$ the left-hand sides of the tight constraints defining F , are linearly independent with the caveat that if we have two tight constraints of the form $\alpha^\top x \leq \beta$ and $-\alpha^\top x \leq -\beta$ we only consider one of these as defining F . For non-degenerate systems, the TDI and near-TDI properties coincide. Namely,

Remark 1.4. Every non-degenerate near-TDI system is TDI.

Proof. Let $Mx \leq b$ be non-degenerate and let $w \in \mathbb{Z}^n$ be admissible weights. Let F denote the set of optimal solutions to $(P : M, b, w)$. Let I be the set of constraints that defines the face F . Let \bar{y} be an optimal solution to $(D : M, b, w)$. By complementary slackness, $\bar{y}_i = 0$ for all $i \notin I$. Since $Mx \leq b$ is non-degenerate, \bar{y} is the unique solution to $M^\top \bar{y} = w$ and $\bar{y}_i = 0$ for all $i \notin I$. As $Mx \leq b$ is near-TDI, it is TD in \mathbb{L}_p and TD in \mathbb{L}_q for distinct primes p and q . It follows that $\bar{y} \in \mathbb{L}_p^m \cap \mathbb{L}_q^m = \mathbb{Z}^m$. Hence, $(D : M, b, w)$ has an integer optimal solution. \square

In the next section, we give a geometric characterization of non-degenerate TDI systems.

1.2 A geometric characterization of non-degenerate TDI systems

Consider a matrix $M \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$. The system $Mx \leq b$ is *resilient* if $Q := \{x \in \mathbb{R}^n : Mx \leq b\}$ is integral and for every $i \in [m] := \{1, 2, \dots, m\}$,

$$Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x \leq b_i - 1\},$$

is also an integral polyhedron. To illustrate this property consider,

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad (1)$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 6 \\ 3 \\ 0 \\ 0 \end{pmatrix}. \quad (2)$$

Denote by Q the polyhedron defined by (1) and by Q' the polyhedron defined by (2), see Figure 2. Both Q and Q' are integral; however, Q is resilient, but Q' is not.

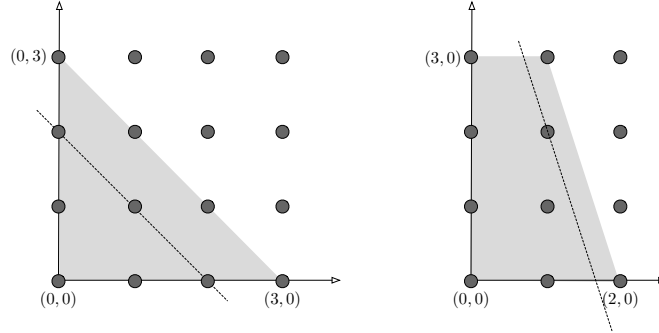


Figure 2: Left Q and a shifted hyperplane, right Q' and a shifted hyperplane.

We prove the following result (see Theorem 4.1 in §4),

Theorem 1.5. *Consider $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Suppose that $\{x \in \mathbb{R}^n : Mx \leq b\}$ is full-dimensional and $Mx \leq b$ is non-degenerate. Then $Mx \leq b$ is TDI if and only if $Mx \leq b$ is resilient.*

For $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ we say that $Mx \leq b$ is a *faceted system* if (i) constraints in $Mx \leq b$ are irredundant (i.e. removing any constraint creates new solutions), (ii) for any constraint, the greatest common divisor (gcd) of the entries on the LHS and RHS is one, and (iii) $\{x \in \mathbb{R}^n : Mx \leq b\}$ is full-dimensional. Recall that for a full-dimensional rational polyhedron P , there is, up to scaling, a unique integral system with irredundant constraints that describes P [6, Corollary 3.31]. It follows that there is a bijection between faceted systems and full-dimensional rational polyhedra. In particular,

faceted systems can be viewed as geometric objects. Consider an integral matrix M and an integral vector b where $Mx \leq b$ is TDI. Suppose that for some constraint i , the gcd of the LHS and RHS is $\alpha \geq 2$. Then, the system obtained from $Mx \leq b$ by dividing constraint i by α , is also TDI. Hence, when studying TDI systems there is no loss of generality in assuming condition (ii) holds. However, conditions (i) and (iii) are actual restrictions.

Next we give a geometric interpretation of resiliency for faceted systems.

Proposition 1.6. *Let $Mx \leq b$ be a faceted system which defines an integral polyhedron Q . Then $Mx \leq b$ is resilient if and only if for every facet F of Q the polyhedron Q_F is integral where Q_F is obtained from Q by shifting-in the supporting hyperplane for F to the next integer lattice point.*

Consider for instance the polyhedron Q described by (1) and illustrated in Figure 2. Let F be the face of Q determined by the supporting hyperplane $H = \{x \in \mathbb{R}^2 : x_1 + x_2 = 3\}$. Shifting-in H (i.e. decreasing the RHS) until the hyperplane contains a new lattice point, yields $H' = \{x \in \mathbb{R}^2 : x_1 + x_2 = 2\}$. Then Q_F is the polyhedron described by (1) where we change $b_1 = 3$ to $b_1 = 2$. Before we proceed with the proof, recall the following immediate consequence of Bezout's lemma,

Remark 1.7. $\alpha^\top x = \beta$ where $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}$ has solution in \mathbb{Z}^n if and only if $\gcd(\alpha_1, \dots, \alpha_n) | \beta$.

Proof of Proposition 1.6. Let m, n denote the number of rows, respectively, columns of M . There exists a bijection between constraints $i \in [m]$ of $Mx \leq b$ and facets F_i of Q where $F_i = Q \cap H_i$ and $H_i = \{x : \text{row}_i(M)x = b_i\}$ is a supporting hyperplane of Q [6, Theorem 3.27].

Pick $i \in [m]$ and let $H'_i = \{x : \text{row}_i(M)x = b_i - 1\}$.

Claim. $H'_i \cap \mathbb{Z}^n \neq \emptyset$.

Proof of Claim. Since Q is integral, the face $F_i \subseteq H_i$ contains an integral point. Hence, $\text{row}_i(M)x = b_i$ has an solution in \mathbb{Z}^n . Remark 1.7 implies that $\mu := \gcd(M_{i1}, \dots, M_{in}) | b_i$. Since $Mx \leq b$ is faceted, we have $\mu = 1$. Therefore, by Remark 1.7 $\text{row}_i(M)x = b_i - 1$ has an integer solution. \diamond

Since there is no integer between $b_i - 1$ and b_i we have $\{x : b_i - 1 < \text{row}_i(M)x < b_i\} \cap \mathbb{Z}^n = \emptyset$. This implies with the claim that H'_i is obtained by shifting-in H_i to the next integer lattice point. \square

Remark 1.4 says that non-degenerate near-TDI systems are TDI. Moreover, if in addition the system is faceted, then we have a geometric description of TDI via Theorem 1.5 and Proposition 1.6. This motivates the following problems,

- Can we characterize when a near-TDI faceted system is TDI?
- Can we find a geometric characterization of when a faceted system is TDI?

In closing, we will observe that each of the inclusions in Figure 1 are strict. Consider the system $Mx \leq b$ described in (2). The associated polyhedron $\{x : Mx \leq b\}$ is integral. However, $Mx \leq b$ is not resilient. Therefore, by Theorem 1.5, $Mx \leq b$ is not TDI. Since $Mx \leq b$ is non-degenerate, it then follows from Remark 1.4 that $Mx \leq b$ is not near-TDI either. Consider now the system,

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (3)$$

with the associated dual problem $(D : M, b, w)$,

$$\min\{0 : 2y_1 + 3y_2 = w, y_1, y_2 \geq 0\}. \quad (4)$$

w admissible iff $w \geq 0$. If w is admissible and not equal to zero, then $x = 0$ is the unique optimal solution for $(P : M, b, w)$. Observe that $\{2y_1 + 3y_2 : y_1, y_2 \in \mathbb{Z}_+\} = \mathbb{Z} \setminus \{1\}$. This implies that (4) has an p -adic solution for every prime p and every choice $w \in \mathbb{Z}_+$ but that (4) does not have an integral solution when $w = 1$. In particular, (3) is near-TDI but not TDI. Note that (3) is not a faceted system, however.

1.3 Totally dyadic systems

A system is totally dual dyadic (abbreviated as TDD) if it is TD in $\mathbb{L}(\{2\})$, i.e. for every integer admissible w , $(D : M, b, w)$ has an optimal solution that is dyadic. We find a sufficient condition for a system of linear inequalities to be TDD by relaxing the resilient condition. Namely, let us say that a system $Mx \leq b$ (where $M \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$) is *half-resilient* if $Q := \{x \in \mathbb{R}^n : Mx \leq b\}$ is integral and for every $i \in [m]$,

$$Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x \leq b_i - 1\} \quad \text{or} \quad Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x \leq b_i - 2\}$$

is also an integral polyhedron. Clearly, every resilient system is half-resilient.

We prove the following result (see Corollary 3.4 part (b)),

Theorem 1.8. *Consider matrix $M \in \mathbb{Z}^{m \times n}$ and vector $b \in \mathbb{Z}^m$. Suppose that $Q = \{x \in \mathbb{R}^n : Mx \leq b\}$ is full-dimensional. If $Mx \leq b$ is half-resilient then $Mx \leq b$ is TDD.*

The study of TDD systems was initially motivated by a conjecture on ideal clutters. A *clutter* \mathcal{C} is a family of sets over ground set $E(\mathcal{C})$ with the property that no distinct pair $S, S' \in \mathcal{C}$ satisfies $S \subseteq S'$. Given a clutter \mathcal{C} , we define the 0, 1 matrix $T(\mathcal{C})$ where the rows of $T(\mathcal{C})$ correspond to the characteristic vectors of the sets in \mathcal{C} . A clutter \mathcal{C} is *ideal* if the polyhedron $\{x \geq \mathbf{0} : T(\mathcal{C})x \geq \mathbf{1}\}$ is integral. Seymour [21, §79.3e], proposed the following conjecture,

The Dyadic Conjecture.

Let \mathcal{C} be an ideal clutter, then $T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is TDD.

This conjecture is known to hold when \mathcal{C} is the clutter of any one of the following families: (a) odd cycles of a graph [15, 11], (b) T -cuts of a graft [17], (c) T -joins of a graft [4], (d) dicuts of a directed graph [18], (e) dijoins of a directed graph [14]. Moreover, when \mathcal{C} is ideal, $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ and $\min\{w^\top x : T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\} = 2$ then the dual has an optimal solution that is dyadic [1]. For both (a) and (b) the system $T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is in fact $\frac{1}{2}$ -TDI. In this paper, we present a result based on the intersection of members of the clutter and its blocker. A set $B \subseteq E(\mathcal{C})$ is a *cover* of \mathcal{C} if $B \cap S \neq \emptyset$ for every $S \in \mathcal{C}$. The set of all inclusion-wise minimal covers of a clutter \mathcal{C} forms another clutter called the *blocker* of \mathcal{C} . We denote by $b(\mathcal{C})$ the blocker of \mathcal{C} .

We prove the following result (see Corollary 5.2(a)),

Theorem 1.9. *Let \mathcal{C} be an ideal clutter. If for every $S \in \mathcal{C}$ and every $B \in b(\mathcal{C})$ we have $|S \cap B| \leq 3$ then $T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is TDD.*

Note, the following related result [8, Theorem 2].

Theorem 1.10. *Let \mathcal{C} be an ideal clutter. If for every $S \in \mathcal{C}$ and every $B \in b(\mathcal{C})$ we have $|S \cap B| \leq 2$ then $T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is TDI.*

1.4 Organization of the paper

In §2 we give a characterization of when a system is TD in \mathbb{L} for a heavy set \mathbb{L} . Since the statement is a bit technical, we postpone the proof of that result until §6 and first focus on applications. In §3 we derive sufficient conditions for a system to be TD in \mathbb{L} for various choices of \mathbb{L} and explain the role of resiliency and half-resiliency. §4 leverages these results to characterize non-degenerate TDI systems. A special case of Seymour's Dyadic Conjecture is proved in §5.

2 The key result

In this section we present a characterization of when a system $Mx \leq b$ is Totally Dual in \mathbb{L} for the case where \mathbb{L} is heavy. First, we need to present the central notion of the *tilt constraints*.

2.1 The tilt constraint

For a subset S of \mathbb{R}^n , $\text{aff}(S)$ denotes the *affine hull* of S , i.e., the smallest affine space that contains S . Consider a system $Mx \leq b$ where M is an $m \times n$ matrix. Constraint $i \in [m]$ of $Mx \leq b$ is *tight* for

some $x \in \mathbb{R}^n$ if $\text{row}_i(M)x = b_i$. Constraint $i \in [m]$ of $Mx \leq b$ is *tight* for some $S \subseteq \mathbb{R}^n$ if it is tight for every $x \in S$. The *implicit equalities* of $Mx \leq b$ are the constraints of $Mx \leq b$ that are tight for $Q := \{x \in \mathbb{R}^n : Mx \leq b\}$. Given a set $S \subseteq \mathbb{R}^n$ we denote by $I_{M,b}(S) \subseteq [m]$ the index set of tight constraints of $Mx \leq b$ for S . We abuse the terminology and say that F is a face of the system $Mx \leq b$ to mean that F is a face of Q . Given faces F, F^+ of Q we say that F^+ is a *down-face* of F , if $F \subset F^+$ and $\dim(F^+) = \dim(F) + 1$.

Consider now nonempty faces F, F^+ of $Mx \leq b$ where F^+ is a down-face of F and assume that F is defined by a supporting hyperplane $\{x \in \mathbb{R}^n : w^\top x = \tau\}$ for some (non-zero) vector $w \in \mathbb{R}^n$ and some $\tau \in \mathbb{R}$, i.e., $F = Q \cap \{x \in \mathbb{R}^n : w^\top x = \tau\}$ and $Q \subseteq \{x \in \mathbb{R}^n : w^\top x \leq \tau\}$. Pick, $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$ and define,

$$\frac{1}{\tau - w^\top \rho} \sum \left([b_i - \text{row}_i(M)\rho] u_i : i \in I_{M,b}(F) \setminus I_{M,b}(F^+) \right) = 1. \quad (5)$$

This is a constraint of the form, $\alpha^\top u = 1$, where α is determined by w, F, F^+, τ, ρ . It defines a hyperplane in the space of the variables u_i . In our applications, we will be given α and will be interested in knowing if there exists $u_i \in \mathbb{L}$ that satisfy $\alpha^\top u = 1$ for some suitable choice of \mathbb{L} . Recall, that F and F^+ are determined by their tight constraints, i.e. $F = Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x = b_i, i \in I_{M,b}(F)\}$ and $F^+ = Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x = b_i, i \in I_{M,b}(F^+)\}$. Therefore, since $F \subset F^+$ we have $I_{M,b}(F) \setminus I_{M,b}(F^+) \neq \emptyset$. It follows that (5) has a least one variable u_i (with a nonzero coefficient). In particular, (5) always has a solution with all variables u_i in \mathbb{R} .

We say that (5) is the (w, F, F^+) -tilt constraint for the system $Mx \leq b$ (we omit specifying $Mx \leq b$ when clear from the context). We do not refer to τ as $w^\top x = \tau$ will be a supporting hyperplane of Q exactly when w is admissible and τ is the optimal value of $(P : M, b, w)$, thus τ is determined by w and $Mx \leq b$. We do not refer to ρ as (5) is independent of the choice of $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$ as we will see in Remark 6.10.

Example 2.1. Consider,

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \tau = 3. \quad (6)$$

Note that $Q := \{x \in \mathbb{R}^n : Mx \leq b\}$ is given in Figure 2. Observe that $Q \cap \{x : w^\top x = \tau\}$ contains a unique point $(0, 3)$ that forms a face F of $Mx \leq b$. Let F_1^+ be the line segment with ends $(0, 3)$ and $(3, 0)$. Let F_2^+ be the line segment with ends $(0, 3)$ and $(0, 0)$. Then for $i = 1, 2$, F_i^+ is a down-face of F . Consider $\rho_1 = (3, 0)$ and $\rho_2 = (0, 0)$. We have for $i = 1, 2$, $\rho_i \in F_i^+ \setminus F$. Note, $I_{M,b}(F) = \{1, 2\}$, $I_{M,b}(F_1^+) = \{1\}$, and $I_{M,b}(F_2^+) = \{2\}$. Then the (w, F, F_1^+) -tilt and (w, F, F_2^+) -tilt constraints are

respectively,

$$\begin{aligned} \frac{1}{\tau - w^\top \rho_1} [b_2 - \text{row}_2(M) \rho_1] u_2 = 1 &\iff \frac{3}{3} u_2 = 1, \\ \frac{1}{\tau - w^\top \rho_2} [b_1 - \text{row}_1(M) \rho_2] u_1 = 1 &\iff \frac{3}{3} u_1 = 1. \end{aligned}$$

2.2 The characterization

Consider a set $S = \{a^1, \dots, a^m\}$ of integer vectors in \mathbb{Z}^n . Then S is an \mathbb{L} -generating set for a cone (\mathbb{L} -GSC) if every vector in the intersection of the conic hull of the vectors in S and \mathbb{L}^n can be expressed as a conic combination of the vectors in S with multipliers in \mathbb{L} [2]. Here is our key result,

Theorem 2.2. *Let $M \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and let \mathbb{L} be a heavy set. Denote by $M^\#x = b^\#$ the implicit equalities of $Mx \leq b$. Then, $Mx \leq b$ is TD in \mathbb{L} if and only if both of the following conditions hold:*

- i. *The rows of $M^\#$ form an \mathbb{L} -GSC;*
- ii. *For every admissible $w \in \mathbb{L}^n$, denote by F the set of optimal solutions of $(P : M, b, w)$ and let F^+ be a down-face of F . Then, the (w, F, F^+) -tilt constraint has a solution with variables in \mathbb{L} .*

The proof is postponed until §6. Note that condition (i) is vacuously true when the polyhedron $\{x : Mx \leq b\}$ is full-dimensional. We have the following analogue of Remark 1.1 for tilt constraints.

Remark 2.3. *Let $\mathbb{L} := \mathbb{L}(S)$ where S is a set of primes. It suffices to consider in condition (ii) of Theorem 2.2, (w, F, F^+) -tilt constraints for $w \in \mathbb{Z}^n$.*

Proof. Consider a (w, F, F^+) -tilt constraint with $w \in \mathbb{L}^n$. For some μ that is a product of integer powers of primes in S we have $\mu w \in \mathbb{Z}^n$. If \bar{u} is a solution for the $(\mu w, F, F^+)$ -tilt constraint with entries in \mathbb{L} , then $\frac{1}{\mu} \bar{u}$ is a solution for the (w, F, F^+) -tilt constraint with entries in \mathbb{L} . \square

We are now ready for our first application of Theorem 2.2.

Proof of Theorem 1.2. Consider an arbitrary (w, F, F^+) -tilt constraint $\alpha^\top u = 1$ of $Mx \leq b$ where $w \in \mathbb{Z}^n$. Because of Theorem 2.2 and Remark 2.3, it suffices to show that $\alpha^\top u = 1$ has a solution in \mathbb{L}_q for every prime q . To that end, we will show that $\alpha^\top u = 1$ has an integral solution. Consider $i \in [k]$. Theorem 2.2 implies that $\alpha^\top u = 1$ has a solution u^i with entries in $\mathbb{L}(\mathcal{S}_i)$. For some μ_i that is a product of integer powers of primes in \mathcal{S}_i we have $\mu_i u^i$ is integral. Since the gcd of μ_1, \dots, μ_k equals 1, by Bezout's lemma we have $s_1, \dots, s_k \in \mathbb{Z}$ for which $\sum_{i \in [k]} s_i \mu_i = 1$. Let $\bar{u} = \sum_{i \in [k]} s_i \mu_i u^i$. Then

$$\alpha^\top \bar{u} = \sum_{i \in [k]} s_i \mu_i \alpha^\top u^i = \sum_{i \in [k]} s_i \mu_i = 1.$$

Thus \bar{u} is an integral solution to $\alpha^\top u = 1$ as required. \square

The reader might have noticed that in the proof Theorem 1.2 it is shown that if an affine space contains points in $\mathbb{L}(\mathcal{S}_i)$ for all $i \in [k]$, then it also contains an integer point. We can apply that same idea to the affine hull of the optimal solutions to $(D: M, b, w)$ and use Theorem 6.1 for an alternate proof.

3 Applications of Theorem 2.2

In this section, we show how Theorem 2.2 yields sufficient conditions for a system to be Totally Dual in \mathbb{L} for various choices of \mathbb{L} . First we need to define *braces*.

3.1 From braces to solutions of tilt constraints

Consider $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Suppose that $Mx \leq b$ has faces F, F^+ where F^+ is a down-face of F . We say that a pair (\hat{i}, ρ) is an (F, F^+) -brace for $Mx \leq b$ if the following conditions hold,

- b1. $\rho \in \mathbb{Z}^n$ and $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$,
- b2. $\hat{i} \in I_{M,b}(F) \setminus I_{M,b}(F^+)$,
- b3. $|b_{\hat{i}} - \text{row}_{\hat{i}}(M)\rho| > 0$.

Informally, (\hat{i}, ρ) certifies that F and F^+ are distinct faces, namely, ρ is in the affine hull of F^+ , but ρ is not in the affine hull of F as it does not satisfy the implicit equality $\text{row}_{\hat{i}}(M)x = b_{\hat{i}}$ of F . The *gap* of brace (\hat{i}, ρ) is defined as $|b_{\hat{i}} - \text{row}_{\hat{i}}(M)\rho|$. Note, that the gap of a brace is a positive integer.

Theorem 2.2 highlights the importance of solutions to tilt constraints. We will see that braces provide a means to finding such solutions. Before we proceed, we require a definition. Let p be a positive integer. We say that a set $\mathbb{L} \subseteq \mathbb{R}$ is *closed under p -division* if for every $\alpha \in \mathbb{L}$ we have $\frac{1}{p}\alpha \in \mathbb{L}$. For instance p -adic rationals are closed under p -division.

Proposition 3.1. *Let $M \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and assume that $\{x \in \mathbb{R}^n : Mx \leq b\}$ is an integral polyhedron. Let \mathbb{L} be a heavy set that is closed under p -division for some positive integer p . Consider $w \in \mathbb{L}^n$ that is admissible for M, b, w . Let F be the optimal face of $(P: M, b, w)$ and let F^+ be a down-face of F . If there exists an (F, F^+) -brace with gap p then the (w, F, F^+) -tilt constraint has a solution with all variables in \mathbb{L} .*

Proof. Denote by (\hat{i}, ρ) the (F, F^+) -brace. Let $I := I_{M,b}(F) \setminus I_{M,b}(F^+)$. By Remark 6.10 the (w, F, F^+) -tilt constraint is independent of the choice of $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$. Thus, we may assume

it is of the form,

$$\frac{1}{\tau - w^\top \rho} \sum_{i \in I} [b_i - \text{row}_i(M)\rho] u_i = 1. \quad (7)$$

Set $u_i := 0$ for all $i \in I$ where $i \neq \hat{i}$ and set,

$$u_{\hat{i}} := \frac{\tau - w^\top \rho}{b_{\hat{i}} - \text{row}_{\hat{i}}(M)\rho}. \quad (8)$$

Then u is a solution to (7). To complete the proof, it suffices to show that $u_{\hat{i}} \in \mathbb{L}$. Since $\{x \in \mathbb{R}^n : Mx \leq b\}$ is integral, F contains a point $\bar{x} \in \mathbb{Z}^n$. Recall that τ is the optimal value of $(P : M, b, w)$, thus we have $\tau = w^\top \bar{x}$. Since $w \in \mathbb{L}^n$ and $\bar{x} \in \mathbb{Z}^n$ we have $\tau \in \mathbb{L}$ (we are using the fact that $(\mathbb{L}, +)$ is a subgroup). Since $\rho \in \mathbb{Z}^n$ we also have $w^\top \rho \in \mathbb{L}$. Hence, the numerator in (8) is in \mathbb{L} . The denominator is the gap of the (F, F^+) -brace, i.e. p . Since, \mathbb{L} is closed under p -division, $u_i \in \mathbb{L}$ as required. \square

3.2 From resiliency to braces

Consider matrix $M \in \mathbb{Z}^{m \times n}$ and vector $b \in \mathbb{Z}^m$. Let p be a positive integer. The system $Mx \leq b$ is said to be $\frac{1}{p}$ -resilient if (i) $Q := \{x \in \mathbb{R}^n : Mx \leq b\}$ is an integral polyhedron and (ii) for every $i \in [m]$,

$$Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x \leq b_i - s(i)\}$$

is also an integral polyhedron for some $s(i) \in [p]$. Hence, $Mx \leq b$ is resilient iff it is 1-resilient and it is half-resilient iff it is $\frac{1}{2}$ -resilient.

Next, we use the notion of $\frac{1}{p}$ -resiliency to obtain braces with gap p .

Proposition 3.2. *Consider $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Let F be a proper face of $Mx \leq b$ and let F^+ be a down-face of F . If $Mx \leq b$ is $\frac{1}{p}$ -resilient, then there exists an (F, F^+) -brace with gap at most p .*

Proof. Since faces of a polyhedron are obtained by setting suitable inequalities to equality, it follows readily that, if $Mx \leq b$ is $\frac{1}{p}$ -resilient for some positive integer p , then so is every face of $Mx \leq b$. We may thus assume that $F^+ = \{x \in \mathbb{R}^n : Mx \leq b\}$. Pick $\hat{i} \in I := I_{M,b}(F) \setminus I_{M,b}(F^+)$. Define the following optimization problem,

$$\kappa := \sup\{b_{\hat{i}} - \text{row}_{\hat{i}}(M)x : x \in F^+\}. \quad (9)$$

By the choice of \hat{i} , observe that $\kappa > 0$. Consider first the case where $\kappa \leq p$. Since $Mx \leq b$ is $\frac{1}{p}$ -resilient, F^+ is integral. Since κ is finite, it follows that (9) attains its maximum at a point $\rho \in F^+ \cap \mathbb{Z}^n$ (since F^+ is integral, every nonempty face of F^+ contains an integral vector). However, then (\hat{i}, ρ) is an (F, F^+) -brace with gap $\kappa \leq p$ as required. Thus, we may assume that $\kappa > p$. Since $Mx \leq b$ is

$\frac{1}{p}$ -resilient there exists integer $s(\hat{i}) \in [p]$ for which $Q := F^+ \cap \{x \in \mathbb{R}^n : \text{row}_{\hat{i}}(M)x \leq b_{\hat{i}} - s(\hat{i})\}$ is an integral polyhedron. As $\kappa > p$ it follows that the face $Q' := Q \cap \{x \in \mathbb{R}^n : \text{row}_{\hat{i}}(M)x = b_{\hat{i}} - s(\hat{i})\}$ of Q is non-empty. Since Q is integral, so is Q' . Hence, Q' contains an integral point ρ . However, then again (\hat{i}, ρ) is an (F, F^+) -brace with gap $s(\hat{i}) \leq p$ as required. \square

3.3 Totally Dual in \mathbb{L} - sufficient conditions

The next result gives sufficient conditions for $Mx \leq b$ to be TD in \mathbb{L} .

Theorem 3.3. *Let $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ and let $\mathbb{L} \subseteq \mathbb{R}$ where \mathbb{L} is a heavy set. Let p denote a positive integer, and assume that \mathbb{L} is closed under q -divisions for all $q \in \{2, \dots, p\}$. Denote by $M^=x = b^=$ the implicit equalities of $Mx \leq b$. Then $Mx \leq b$ is TD in \mathbb{L} , if both of the following conditions hold,*

- i. The rows of $M^=$ form an \mathbb{L} -GSC.*
- ii. $Mx \leq b$ is $\frac{1}{p}$ -resilient.*

Proof. Consider an admissible $w \in \mathbb{L}^n$, denote by F the set of optimal solutions of $(P: M, b, w)$ and let F^+ be a down-face of F . Because of Theorem 2.2, it suffices to show that the (w, F, F^+) -tilt constraint has a solution with variables in \mathbb{L} . Since $Mx \leq b$ is $\frac{1}{p}$ -resilient, there exists by Proposition 3.2 an (F, F^+) -brace (\hat{i}, ρ) with gap $q \leq p$. By hypothesis, \mathbb{L} is closed under q -division, and $\{x \in \mathbb{R}^n : Mx \leq b\}$ is an integral polyhedron, as $Mx \leq b$ is $\frac{1}{p}$ -resilient. It then follows from Proposition 3.1 that the (w, F, F^+) -tilt constraint has a solution in \mathbb{L} as required. \square

Given a prime p , let $p_1, \dots, p_\ell := p$ denote the set of all primes up to and including p . We write $\mathbb{L}_{[p]}$ for $\mathbb{L}(\{p_1, \dots, p_\ell\})$. In particular, $\mathbb{L}_{[p]}$ is a heavy set. We have the following immediate consequences of Theorem 3.3,

Corollary 3.4. *Let $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ and suppose that $\{x \in \mathbb{R}^n : Mx \leq b\}$ is full-dimensional.*

- (a) If $Mx \leq b$ is resilient, then it is TD in \mathbb{L} for every heavy set \mathbb{L} .*
- (b) If $Mx \leq b$ is half-resilient, then it is TDD.*
- (c) If $Mx \leq b$ is $\frac{1}{p}$ -resilient for an integer $p \geq 2$, then it is TD in $\mathbb{L}_{[p]}$.*

Proof. Hypothesis (i) in Theorem 3.3 trivially holds as $\{x \in \mathbb{R}^n : Mx \leq b\}$ is full-dimensional. Applying Theorem 3.3 for the case where: $p = 1$ yields (a); $p = 2$ yields (b) as \mathbb{L}_2 is closed under 2-divisions; and $p \geq 3$ yields (c) as $\mathbb{L}_{[p]}$ is closed under q -divisions for every $q \in \{2, \dots, p\}$. \square

We say that $Mx \leq b$ is *p-small* if (i) $\{x \in \mathbb{R}^n : Mx \leq b\}$ is an integral polytope and (ii) for every $i \in [m]$ and every extreme point $\bar{x} \in Q$ we have $\text{row}_i(M)\bar{x} \geq b_i - p$. Equivalently (ii) holds when the largest entry of the *slack matrix*¹ of $Mx \leq b$ is at most p .

Remark 3.5. *If $Mx \leq b$ is p-small, then $Mx \leq b$ is $\frac{1}{p}$ -resilient.*

Proof. Suppose $Mx \leq b$ is *p-small*. Pick $i \in [m]$ where $\text{row}_i(M)x = b_i$ is not an implicit equality of $\{x \in \mathbb{R}^n : Mx \leq b\}$. The optimal solution to,

$$s(i) := \max\{b_i - \text{row}_i(M)x : Mx \leq b\},$$

is attained by an extreme point of $Q = \{x : Mx \leq b\}$ and $s(i) \leq p$. Then $Q \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x \leq b_i - s(i)\}$ is a face of Q and hence is integral. By definition, this implies that $Mx \leq b$ is $\frac{1}{p}$ -resilient. \square

Combining this observation with Corollary 3.4 we deduce that for $M \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $\{x : Mx \leq b\}$ full-dimensional, when $Mx \leq b$ is 2-small it is TDD.

4 Total Dual Integrality

In this section, we give a geometric characterization of non-degenerate TDI systems. Note that a \mathbb{Z} -GSC is also known as a *Hilbert Cone* [13]. We are ready to state the main result of this section.

Theorem 4.1. *Let $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ where $Mx \leq b$ is non-degenerate. Let $M^\#x = b^\#$ denote the implicit equalities of $Mx \leq b$. Then $Mx \leq b$ is TDI if and only if both the following conditions hold:*

- i. The rows of $M^\#$ form a Hilbert Cone;*
- ii. $Mx \leq b$ is resilient.*

Observe that when $\{x \in \mathbb{R}^n : Mx \leq b\}$ is full-dimensional there are no implicit constraints and (i) holds trivially. It follows that Theorem 1.5 is an immediate consequence of Theorem 4.1. In §4.1 we show that (i) and (ii) imply that $Mx \leq b$ is TDI. Finally, in §4.2 we show that $Mx \leq b$ TDI implies that (i) and (ii) hold.

¹see [6, §4.10] for a definition.

4.1 Sufficiency for TDI

Restricting Theorem 3.3 to the case $p = 1$ yields,

Corollary 4.2. *Let $M \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ and let $\mathbb{L} \subseteq \mathbb{R}$ where \mathbb{L} is a heavy set. Let $M^\#x = b^\#$ denote the implicit equalities of $Mx \leq b$. Then $Mx \leq b$ is TD in \mathbb{L} if the following conditions hold,*

- i. The rows of $M^\#$ form an \mathbb{L} -GSC;*
- ii. $Mx \leq b$ is resilient.*

We can now prove the promised result.

Proposition 4.3. *In Theorem 4.1 conditions (i) and (ii) imply $Mx \leq b$ is TDI.*

Proof. Let p be an arbitrary prime and pick $w \in \mathbb{L}_p^n$ that is a conic combination of the rows in $M^\#$. Then for some integer $k \geq 0$, $p^k w \in \mathbb{Z}^n$. It follows by (i) in Theorem 4.1 that $p^k w$ can be expressed as an integer conic combination of the rows in $M^\#$. Hence, w can be expressed as a p -adic conic combination of the rows in $M^\#$. In particular, (i) holds in Corollary 4.2 for $\mathbb{L} = \mathbb{L}_p$. We can thus apply Corollary 4.2 and deduce that $Mx \leq b$ TD in \mathbb{L}_p . As p was an arbitrary prime, $Mx \leq b$ is near-TDI and the result follows from Remark 1.4. \square

4.2 Necessity for TDI

A system $Mx \leq b$ is TDI if for every face F , the left hand side of the tight constraints for F form a Hilbert Cone [13]. It follows that condition (i) in Theorem 4.1 is necessary for $Mx \leq b$ to be TDI. The goal of this section is to prove the next result (thereby completing the proof of Theorem 4.1).

Proposition 4.4. *In Theorem 4.1 $Mx \leq b$ TDI implies $Mx \leq b$ is resilient.*

To that end, we first require a definition. Let $M \in \mathbb{Z}^{m \times n}$ and let $b, b' \in \mathbb{Z}^m$. Let F and F' be faces of $Mx \leq b$ and of $Mx \leq b'$, respectively. We say that F and F' are *mates* if $I_{M,b}(F) = I_{M,b'}(F')$, i.e. the index sets of the tight constraints for F and F' are the same. We denote by e^i the vector with $e^i_i = 1$ and $e^i_j = 0$ for all $j \neq i$. For a polyhedron $Q := \{x \in \mathbb{R}^n : Mx \leq b\}$ the *lineality space* of Q is denoted, $\text{lin}(Q)$.

Proposition 4.5. *Let $M \in \mathbb{Z}^{m \times n}$ and let $b, b' \in \mathbb{Z}^m$ where $b' = b - e^i$ for some $i \in [m]$. Suppose that $P := \{x \in \mathbb{R}^n : Mx \leq b\}$ is integral and $Mx \leq b$ is non-degenerate. Then, every minimal face F' of $Mx \leq b'$ that is not a face of $Mx \leq b$ has a mate F of $Mx \leq b$.*

Proof. Let $P' = \{x \in \mathbb{R}^n : Mx \leq b'\}$. Note that $\text{lin}(P) = \{x \in \mathbb{R}^n : Mx = \mathbf{0}\} = \text{lin}(P')$. There exist pointed polyhedra R and R' with the property that $P = R + \text{lin}(P)$ and $P' = R' + \text{lin}(P')$. Thus, minimal faces of P and P' have the same dimension $d := \dim(\{x \in \mathbb{R}^n : Mx = \mathbf{0}\})$. Let $\alpha := \text{row}_i(M)$ and let $\beta := b_i$. Note, that $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}$.

Claim. $S := \{x \in \mathbb{R}^n : \beta - 1 < \alpha^\top x < \beta\} \cap \mathbb{Z}^n = \emptyset$.

Proof of Claim. If $\bar{x} \in S \cap \mathbb{Z}^n$ then $\alpha^\top \bar{x} \in \mathbb{Z}$, but $\beta - 1 < \alpha^\top \bar{x} < \beta$, a contradiction. \diamond

Let F' be a minimal face of P' that is not a face of P . We will show that F' has a mate F of $Mx \leq b$. Since F' is not a face of P , it is the intersection of a face L of dimension $d+1$ of P and the hyperplane $\{x \in \mathbb{R}^n : \alpha^\top x = \beta - 1\}$. Hence, $L \cap \{x \in \mathbb{R}^n : \alpha^\top x < \beta - 1\}$ and $L \cap \{x \in \mathbb{R}^n : \alpha^\top x > \beta - 1\}$ are non-empty. L corresponds to a face L_q of dimension 1 of R where $L = L_q + \text{lin}(P)$. Then $L_q \cap \{x \in \mathbb{R}^n : \alpha^\top x < \beta - 1\}$ and $L_q \cap \{x \in \mathbb{R}^n : \alpha^\top x > \beta - 1\}$ are non-empty. Since $\alpha^\top x \leq \beta$ is not a redundant constraint of $Mx \leq b$, it follows that L_q is a line segment or a half-line with an end v_q where $\alpha^\top v_q > \beta - 1$. Then v_q is an extreme point of R , hence $F := \{v_q\} + \text{lin}(P)$ is a minimal face of P . Since P is integral F contains an integral point v in G . It follows from the Claim that $\alpha^\top v = \beta$. Thus, F is defined by the tight constraints of L and $\alpha^\top x = \beta$. Since F' is defined by the tight constraints of L and $\alpha^\top x = \beta - 1$, F is the mate of F' . \square

Consider a set $S = \{a^1, \dots, a^m\}$ of vectors in \mathbb{Z}^n . S is an \mathbb{L} -generating set for a subspace (\mathbb{L} -GSS) if every vector in the intersection of the linear hull of the vectors in S and \mathbb{L}^n can be expressed as a linear combination of the vectors in S with multipliers in \mathbb{L} .

Proposition 4.4. Since b is integral and $Mx \leq b$ is TDI, $\{x \in \mathbb{R}^n : Mx \leq b\}$ is an integral polyhedron. Now pick $i \in [m]$ and let $b' := b - e^i$. We need to show that $\{x \in \mathbb{R}^n : Mx \leq b'\}$ is an integral polyhedron. Consider an arbitrary minimal face F' . By Proposition 4.5, we may assume $\{x \in \mathbb{R}^n : Mx \leq b\}$ has a minimal face F that is a mate of F' . Denote by $Nx = f$ the tight constraints of $Mx \leq b$ for F . Since $Mx \leq b$ is TDI, the rows of N form a Hilbert basis [13], or equivalently, a \mathbb{Z} -GSC. It is proved in [2], that every \mathbb{Z} -GSC is an \mathbb{Z} -GSS and that if the rows of a matrix form a \mathbb{Z} -GSS, then so do the columns (see Proposition 6.3 and Proposition 6.4 in §6 for details). It follows that the columns of N form a \mathbb{Z} -GSS. Minimal faces are affine spaces. Thus $F' = \{x \in \mathbb{R}^n : Nx = f'\}$ where $Nx = f'$ denotes the tight constraints for F' of $Mx \leq b'$. Since $b' \in \mathbb{Z}^m$, and since the columns of N form a \mathbb{Z} -GSS, there exists an integral solution x' to $Nx = f'$. Then x' is an integral point of F' . We proved that every minimal face of $\{x \in \mathbb{R}^n : Mx \leq b'\}$ has an integral point, i.e., that it is an integral polyhedron as required. \square

5 The Dyadic Conjecture

Define $\mathcal{P} := \{0\} \cup \{2^i : i \in \mathbb{Z}_+\}$. We prove,

Theorem 5.1. *Let \mathcal{C} be an ideal clutter. If for all $S \in \mathcal{C}$ and $B \in b(\mathcal{C})$ we have $|S \cap B| - 1 \in \mathcal{P}$ then $T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is TDD.*

A clutter \mathcal{C} is said to be *binary* if for every $S \in \mathcal{C}$ and every $B \in b(\mathcal{C})$ we have $|S \cap B|$ odd. See [22] for a characterization. Observe that Theorem 5.1 implies,

Corollary 5.2. *Let \mathcal{C} be an ideal clutter. Then $T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is TDD if either (a) for all $S \in \mathcal{C}$ and $B \in b(\mathcal{C})$, we have $|S \cap B| \leq 3$, or (b) \mathcal{C} is a binary clutter and for all $S \in \mathcal{C}$ and $B \in b(\mathcal{C})$, we have $|S \cap B| \leq 5$.*

The proof of Theorem 5.1 relies on Theorem 2.2. We therefore need to restate the set covering linear program and its dual in the setting of the linear programs $(P: M, b, w)$ and $(D: M, b, w)$. To that effect, we assume when we consider a clutter \mathcal{C} that the number of members of \mathcal{C} is m and that the size of the ground set is n . Recall that for a clutter \mathcal{C} , $T(\mathcal{C})$ denotes the 0, 1 matrix with rows corresponding to characteristic vectors of \mathcal{C} . Thus $T(\mathcal{C})$ is an $m \times n$ matrix. Define the matrix,

$$M(\mathcal{C}) := \begin{pmatrix} -T(\mathcal{C}) \\ -I_n \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix. In addition, we define vector $d(\mathcal{C}) \in \{-1, 0\}^{m+n}$ where $d(\mathcal{C})_i = -1$ for $i = 1, \dots, m$ and $d(\mathcal{C})_i = 0$ for $i = m + 1, \dots, n$. Then the primal-dual pair of set covering linear programs,

$$\begin{aligned} \min\{w^\top x : T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\} \\ \max\{\mathbf{1}^\top y : T(\mathcal{C})^\top y \leq w, y \geq \mathbf{0}\}, \end{aligned}$$

can be expressed as $(P: M(\mathcal{C}), d(\mathcal{C}), -w)$ and $(D: M(\mathcal{C}), d(\mathcal{C}), -w)$ respectively.

We leave the following observation as an easy exercise.

Remark 5.3. *$T(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$ is TDD if and only if $M(\mathcal{C})x \leq d(\mathcal{C})$ is TDD.*

Denote by $\text{rec}(R)$ the recession cone of a polyhedron R . A polyhedral cone $C \subseteq \mathbb{R}^n$ is *generated* by a set S of vectors in \mathbb{R}^n if C is equal to the conic hull of the vectors in S . We require the following observation.

Remark 5.4. *Let P be a polyhedron and suppose that $\text{rec}(P)$ is generated by a finite set S of vectors. Let P' be a face of P then (a) $\text{rec}(P')$ is a face of $\text{rec}(P)$, and (b) $\text{rec}(P')$ is generated by a subset S' of S .*

Proof. Since P is a polyhedron, then $P = \{x : Mx \leq b\}$ for some matrix M and vector b . **(a)** Let $I := I_{M,b}(P')$, then $P' = P \cap \{x : \text{row}_i(M)x = b_i, i \in I\}$. We have $\text{rec}(P) = \{x : Mx \leq \mathbf{0}\}$ [6, Proposition 3.15]. Then observe that $\text{rec}(P') = \text{rec}(P) \cap \{x : \text{row}_i(M)x = 0, i \in I\}$. **(b)** $\text{rec}(P')$ is generated by its extreme rays. Observe that every extreme ray in the face of a polyhedral cone is an extreme ray in the polyhedral cone and use (a). \square

We are now ready for the last proof of this section.

Theorem 5.1. Recall that \mathbb{L}_2 denotes the set of dyadic rationals. Let $M := M(\mathcal{C})$, $b := d(\mathcal{C})$, $n = |E(\mathcal{C})|$ and $m = |\mathcal{C}|$. In light of Remark 5.3 we need to prove that $Mx \leq b$ is TDD. It suffices to prove that conditions (i) and (ii) in Theorem 2.2 hold. Note, that (i) trivially holds as $\{x : Mx \leq b\}$ is full-dimensional. Thus it suffices to prove (ii). To that end, consider an admissible $w \in \mathbb{L}_2^n$ of $(P : M, b, w)$, denote by F the set of optimal solutions, and let F^+ be a down-face of F . We need to show that the (w, F, F^+) -tilt constraint has a solution with variables in \mathbb{L}_2 . Because \mathcal{C} is ideal, $Q := \{x : T(\mathcal{C}) \geq \mathbf{1}, x \geq \mathbf{0}\} = \{x : Mx \leq b\}$ is integral. Observe that \mathbb{L}_2 is closed under 2^k divisions for any $k \in \mathbb{Z}_+$. Therefore, by Proposition 3.1, it suffices to show there exists an (F, F^+) -brace with gap 2^k for some $k \in \mathbb{Z}_+$.

Case 1. $F^+ \setminus F$ contains an extreme point ρ of Q .

Because \mathcal{C} is ideal, ρ is the characteristic vector of a member B of $b(\mathcal{C})$ [7, Remark 1.16]. Let $\hat{i} \in I_{M,b}(F) \setminus I_{M,b}(F^+)$. Consider first the case where $\hat{i} \in [m]$. Constraint \hat{i} of $Mx \leq b$ says that $\sum_{i \in S} x_i \geq 1$ for some $S \in \mathcal{C}$. Then

$$b_{\hat{i}} - \text{row}_{\hat{i}}(M)\rho = \sum_{i \in S} \rho_i - 1 = |S \cap B| - 1 \in \mathcal{P},$$

where the membership follows by hypothesis. Hence, (\hat{i}, ρ) is the required (F, F^+) -brace and we may assume $\hat{i} \in [m+n] \setminus [m]$. Then constraint \hat{i} of $Mx \leq b$ is $x_{\hat{i}-m} \geq 0$. Therefore, there exists $\rho' \in F^+ \setminus F$ with $\rho'_{\hat{i}-m} > 0$. Since $\text{rec}(Q) = \mathbb{R}_+^n$, it is generated by $S = \{e^1, \dots, e^n\}$. It follows from Remark 5.4 that $\text{rec}(F^+)$ is generated by $S' \subseteq S$. Therefore, ρ' is obtained as the sum of a convex combination of extreme points of F^+ and a conic combination of S' . As Q is integral, so is F^+ . It follows, that we must have $\rho'' \in F^+ \cap \mathbb{Z}^n$ with $\rho''_{\hat{i}} = 1$. Then

$$b_{\hat{i}} - \text{row}_{\hat{i}}(M)\rho'' = \rho''_{\hat{i}-m} = 1,$$

and (\hat{i}, ρ'') is the required (F, F^+) -brace.

Case 2. $F^+ \setminus F$ contains no extreme point.

Since we are in case 2 and $F^+ \supset F$ we have $\text{rec}(F^+) \supset \text{rec}(F)$. It follows from Remark 5.4 that there must be a generator e^j in of $\text{rec}(F^+)$ that is not a generator of $\text{rec}(F)$. Hence, for some integral point $v \in F$ we have $\rho := v + e^j \in F^+ \setminus F$. Let \hat{i} denote the index of a constraint of $Mx \leq b$ that is tight for v but not for ρ . Then (\hat{i}, ρ) is again the required (F, F^+) -brace. \square

6 The proof of Theorem 2.2

In this section we give the proof of Theorem 2.2 (our characterization of systems of inequalities that are TD in \mathbb{L} for a heavy set \mathbb{L}). Along the way, we will also prove Remark 6.10. This section is organized as follows: in §6.1 we explain the role of density; §6.2 reviews results on generating sets and cones; we derive applications of complementary slackness in §6.3; a geometric interpretation of the tilt constraints and the proof of Remark 6.10 is found in §6.4; finally, in §6.5 we present the proof of Theorem 2.2.

6.1 Density and a theorem of the alternative

A key property of heavy sets is density. This allows us to use the following powerful result [2, 3].

Theorem 6.1. *Let \mathbb{L} be a heavy set. Consider a nonempty convex set S , where $\text{aff}(S)$ is a translate of a rational subspace. Then $S \cap \mathbb{L}^n \neq \emptyset$ if and only if $\text{aff}(S) \cap \mathbb{L}^n \neq \emptyset$.*

Based on this result, it is shown in [3] that the problems of checking if a polyhedron contains a dyadic, or p -adic]] can be answered in polynomial time. Contrast this with the NP-hardness of checking if a polyhedron contains an integral point (see, for instance, [5, Chapter 2]).

We require the following theorem of the alternative from [3, Theorem 2.7].

Theorem 6.2. *Suppose $(\mathbb{L}, +)$ forms a subgroup of $(\mathbb{R}, +)$ and $\mathbb{L} \neq \mathbb{R}$. Let $A \in \mathbb{Z}^{m \times n}$ and let $b \in \mathbb{Z}^m$. Then exactly one of the following holds,*

- a. $Ax = b$ has a solution in \mathbb{L}^n ,
- b. there exists $u \in \mathbb{R}^m$ such that $A^\top u \in \mathbb{Z}^n$ and $b^\top u \notin \mathbb{L}$.

6.2 Generating sets and cones

Recall the definition of \mathbb{L} -GSC in §2.2 and of \mathbb{L} -GSS in §4.2. The following was proved for the case where \mathbb{L} is the set of dyadic rationals in [2, Proposition 3.3]. The same proof extends to the following general context, we include it for completeness.

Proposition 6.3. *Suppose $(\mathbb{L}, +)$ forms a subgroup of $(\mathbb{R}, +)$ and $\mathbb{L} \supset \mathbb{Z}$. If $\{a^1, \dots, a^m\} \subset \mathbb{Z}^n$ is an \mathbb{L} -GSC then it is an \mathbb{L} -GSS.*

In [2], it was observed that the converse does not hold.

Proposition 6.3. Let $A \in \mathbb{Z}^{m \times n}$ be the matrix whose columns are a^1, \dots, a^n . Take $b \in \mathbb{L}^m$ such that $A\bar{x} = b$ for some $\bar{x} \in \mathbb{R}^n$. We need to show that the system $Ax = b$ has a solution in \mathbb{L}^n . To this end, let $\bar{x}' := \bar{x} - \lfloor \bar{x} \rfloor \geq \mathbf{0}$ and $b' := A\bar{x}' = b - A\lfloor \bar{x} \rfloor$. Since $A\lfloor \bar{x} \rfloor \in \mathbb{Z}^m$, $\mathbb{L} \supset \mathbb{Z}$, and $(\mathbb{L}, +)$ forms a group, $b' \in \mathbb{L}^m$. By construction, $Ax = b', x \geq \mathbf{0}$ has a solution, namely \bar{x}' . So it has a solution, say $\bar{z}' \in \mathbb{L}^n$, as the columns of A form an \mathbb{L} -GSC. Let $\bar{z} := \bar{z}' + \lfloor \bar{x} \rfloor$, which is also in \mathbb{L}^n since $\mathbb{L} \supset \mathbb{Z}$ and $(\mathbb{L}, +)$ forms a group. Then, $A\bar{z} = A\bar{z}' + A\lfloor \bar{x} \rfloor = b' + A\lfloor \bar{x} \rfloor = b$, so $\bar{z} \in \mathbb{L}^n$ is a solution to $Ax = b$, as required. \square

The next result was proved for the case of p -adic rationals in [2]. The same proof also works for the case of integers. We include it for completeness.

Proposition 6.4. *The following are equivalent for a matrix $A \in \mathbb{Z}^{m \times n}$,*

- a. the rows of A form an \mathbb{Z} -GSS,*
- b. the columns of A form an \mathbb{Z} -GSS,*
- c. whenever $u^\top A$ and Ax are integral, then $u^\top Ax \in \mathbb{Z}$.*

Proof. **(b) \Rightarrow (c)** Choose x and u such that $u^\top A$ and Ax are integral. Let $b = Ax \in \mathbb{Z}^m$. By (b), there exists $\bar{x} \in \mathbb{Z}^n$ such that $b = A\bar{x}$. Thus, $u^\top Ax = u^\top A\bar{x} = (u^\top A)\bar{x} \in \mathbb{Z}$, since $u^\top A \in \mathbb{Z}^m$ and $\bar{x} \in \mathbb{Z}^n$. **(c) \Rightarrow (b)** Pick $b \in \mathbb{Z}^m$ such that $Ax = b$ for some $x \in \mathbb{R}^n$. We need to show that $Ax = b$ has a solution in \mathbb{Z} . Pick any $u \in \mathbb{R}^m$ for which $u^\top A$ is integral. Then by (c), $u^\top b = u^\top Ax \in \mathbb{Z}$. Since this holds for all u such that $u^\top A$ is integral, it then follows from Theorem 6.2 that $Ax = b$ has a solution in \mathbb{Z} . **(a) \Leftrightarrow (c)** Observe that (c) holds for A if and only if it holds for A^\top . Therefore, (b) \Leftrightarrow (c) implies that (a) \Leftrightarrow (c). \square

6.3 Affine hull

Recall that for a non-empty face F of $\{x \in \mathbb{R}^n : Mx \leq b\}$ we have that $F = \{x \in \mathbb{R}^n : Mx \leq b\} \cap \{x \in \mathbb{R}^n : \text{row}_i(M)x = b_i, i \in I_{M,b}(F)\}$ [6, Theorem 3.24]. The affine hull of a polyhedron is characterized by its implicit equalities [6, Section 3.7], namely,

Proposition 6.5. *Let $M^\#x = b^\#$ be the implicit equalities of $Mx \leq b$. Then*

$$\text{aff}(\{x \in \mathbb{R}^n : Mx \leq b\}) = \{x \in \mathbb{R}^n : M^\#x = b^\#\}.$$

In order to use Theorem 6.1 for our purposes, we will need to get an explicit description of the affine hull of the optimal solutions to $(D : M, b, w)$. First, let us state the Complementary Slackness conditions for the primal-dual pair $(P : M, b, w)$ and $(D : M, b, w)$.

$$\text{For all row indices } i \text{ of } M: \text{row}_i(M)x = b_i \text{ or } y_i = 0. \quad (10)$$

Recall for linear programming that a pair of primal, dual feasible solutions are both optimal if and only if complementary slackness holds. We use complementary slackness to characterize the optimal solutions of $(P : M, b, w)$.

Proposition 6.6. *A nonempty face F of $\{x \in \mathbb{R}^n : Mx \leq b\}$ is contained in the optimal solutions of $(P : M, b, w)$ if and only if w is a conic combination² of $S := \{\text{row}_i(M) : i \in I_{M,b}(F)\}$.*

Proof. Let F be a nonempty face of $\{x \in \mathbb{R}^n : Mx \leq b\}$ and suppose that w is a conic combination of S . Then there exists $y \geq \mathbf{0}$ with $w = M^\top y$ and $y_i = 0$ for all $i \notin I_{M,b}(F)$. Then y is feasible for $(D : M, b, w)$ and any $x \in F$ and y satisfy (10). Hence every $x \in F$ is an optimal solution of $(P : M, b, w)$. Conversely, assume every $x \in F$ is an optimal solution of $(P : M, b, w)$. Pick x in the relative interior of F and let y be any optimal solution of $(D : M, b, w)$. Then $y \geq \mathbf{0}$, $w = M^\top y$ by feasibility, and $y_i = 0$ for all $i \notin I_{M,b}(F)$ by (10). It follows that w is a conic combination of S . \square

Next, we use complementary slackness and strict complementarity to characterize the optimal solutions of $(D : M, b, w)$.

Proposition 6.7. *Let F be the set of optimal solutions to $(P : M, b, w)$ where $F \neq \emptyset$. Let y be a feasible solution to $(D : M, b, w)$. Then, y is optimal if and only if $y_i = 0$ for all $i \notin I_{M,b}(F)$. Moreover, there exists an optimal solution y with $y_i > 0$ for all $i \in I_{M,b}(F)$.*

Proof. Suppose that y is optimal. Let $i \notin I_{M,b}(F)$, then by definition of $I_{M,b}(F)$, there exists $x^i \in F$ with $\text{row}_i(M)x^i < b_i$. Since x^i is optimal for $(P : M, b, w)$ it follows from (10) for the pair x^i, y that $y_i = 0$. Suppose now that $y_i = 0$ for all $i \notin I_{M,b}(F)$. Pick any $\bar{x} \in F$, then \bar{x}, y satisfy (10). Hence, y is optimal. Finally, since $(P : M, b, w)$ has an optimal solution, by strict complementarity theorem, there exist primal and dual solutions x and y , respectively which are strictly complementary. Then, $y_i > 0$ for all $i \in I_{M,b}(F)$, as desired. \square

Next, we can characterize the affine hull of optimal solutions of $(D : M, b, w)$.

Proposition 6.8. *Let F be the set of optimal solutions to $(P : M, b, w)$ and assume that $F \neq \emptyset$. Then the affine hull of the optimal dual solutions of $(D : M, b, w)$ is given by,*

$$\left\{ y \in \mathbb{R}^m : M^\top y = w, y_i = 0, i \notin I_{M,b}(F) \right\}. \quad (11)$$

²We interpret $\text{cone}(\emptyset) := \{\mathbf{0}\}$.

Proof. By Proposition 6.7 the optimal solutions to $(D: M, b, w)$ are exactly the points in,

$$Q = \left\{ y \in \mathbb{R}^m : M^\top y = w, y \geq \mathbf{0}, y_i = 0, i \notin I_{M,b}(F) \right\}.$$

By the "moreover" part of Proposition 6.7 $y_i \geq 0$ is not a tight constraint for Q when $i \notin I_{M,b}(F)$. The result now follows from Proposition 6.5. \square

6.4 A Geometric interpretation

Next, we provide a geometric interpretation of the tilt constraints.

Proposition 6.9. *Consider $Mx \leq b$ with nonempty faces F, F^+ where F^+ is a down-face of F and where F is defined by a supporting hyperplane $\{x \in \mathbb{R}^n : w^\top x = \tau\}$. Let $I := I_{M,b}(F) \setminus I_{M,b}(F^+) \neq \emptyset$ and assume that for every $i \in I$ we are given $u_i \in \mathbb{R}$. Define,*

$$\bar{w}^\top := w^\top - \sum_{i \in I} u_i \text{row}_i(M) \quad \text{and} \quad \bar{\tau} := \tau - \sum_{i \in I} u_i b_i. \quad (12)$$

Then the following are equivalent,

- a. $F^+ \subseteq \{x \in \mathbb{R}^n : \bar{w}^\top x = \bar{\tau}\}$,
- b. $\bar{w} \in \text{span}\{\text{row}_i(M) : i \in I_{M,b}(F^+)\}$,
- c. *the (w, F, F^+) -tilt constraint for $Mx \leq b$ is satisfied by $u_i : i \in I$.*

Example 2.1 - continued. Recall, M, b, w, τ as in (6). We had defined, faces $F = \{(0, 3)\}$ and F_1^+ which consists of the line segment with ends $(0, 3)$ and $(3, 0)$. Recall that $\rho_1 = (3, 0)$. The (w, F, F_1^+) -tilt constraint has a unique solution $u_2 = 1$. Let $\bar{w} = w - u_2 \text{row}_2(M) = (0, 1) - (-1, 0) = (1, 1)$ and let $\bar{\tau} = \tau - u_2 b_2 = 3$. Then (c) holds in Proposition 6.9 and the reader can verify that $F_1^+ \subseteq \{x \in \mathbb{R}^n : \bar{w}^\top x = \bar{\tau}\}$ and that $\bar{w}^\top \in \text{span}\{\text{row}_1(M)\}$.

Proposition 6.9. Let $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$ and $H = \{x \in \mathbb{R}^n : \bar{w}^\top x = \bar{\tau}\}$.

Claim 1. $F \subseteq \{x \in \mathbb{R}^n : \bar{w}^\top x = \bar{\tau}\}$.

Proof of Claim. Let $x \in F$. Then $w^\top x = \tau$ and for all $i \in I \subseteq I_{M,b}(F)$, $\text{row}_i(M)x = b_i$. Thus,

$$\bar{w}^\top x = \left[w^\top - \sum_{i \in I} u_i \text{row}_i(M) \right] x = \tau - \sum_{i \in I} u_i b_i = \bar{\tau},$$

hence for every $x \in F$, $\bar{w}^\top x = \bar{\tau}$. \diamond

Claim 2. $\bar{w}^\top \rho = \bar{\tau}$ if and only if (c) holds.

Proof of Claim. $\bar{\tau} = \bar{w}^\top \rho$ can be rewritten as,

$$\tau - \sum_{i \in I} u_i b_i = w^\top \rho - \sum_{i \in I} u_i \text{row}_i(M) \rho.$$

Then, observe that this is the (w, F^+, F) -tilt constraint (with the terms rearranged). \diamond

$\boxed{\text{(a)} \Rightarrow \text{(c)}}$ Since (a) holds, $\text{aff}(F^+) \subseteq H$, hence $\bar{w}^\top \rho = \bar{\tau}$. Then (c) holds by Claim 2. $\boxed{\text{(c)} \Rightarrow \text{(a)}}$ By Claim 1 and Claim 2 we have $\text{aff}(F \cup \{\rho\}) \subseteq H$. Since $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$ and $\dim(F^+) = \dim(F) + 1$ we have, $\text{aff}(F \cup \{\rho\}) = \text{aff}(F^+)$. Hence, $\text{aff}(F^+) \subseteq H$ and (a) holds. $\boxed{\text{(a)} \Rightarrow \text{(b)}}$ Since $F^+ \subseteq H$, we have $\text{aff}(F^+) - \rho \subseteq H - \rho$ ³. By Proposition 6.5, $\text{aff}(F^+) = \{x \in \mathbb{R}^n : \text{row}_i(M)x = b_i, i \in I_{M,b}(F^+)\}$. Thus $\text{aff}(F^+) - \rho = \{x \in \mathbb{R}^n : \text{row}_i(M)x = 0, i \in I_{M,b}(F^+)\}$ and similarly $H - \rho = \{x \in \mathbb{R}^n : \bar{w}^\top x = 0\}$. Therefore,

$$\{x \in \mathbb{R}^n : \text{row}_i(M)x = 0, i \in I_{M,b}(F^+)\} \subseteq \{x \in \mathbb{R}^n : \bar{w}^\top x = 0\}. \quad (\star)$$

Taking the orthogonal complement yields,

$$\text{span}\{\text{row}_i(A) : i \in I_{M,b}(F^+)\} \supseteq \text{span}(\bar{w}^\top). \quad (\dagger)$$

Hence, (b) holds. $\boxed{\text{(b)} \Rightarrow \text{(a)}}$ If (b) holds then so does (\dagger) and in turn, so does (\star) . Equivalently, $\text{aff}(F^+) - \rho \subseteq H - \rho$. Hence, $\text{aff}(F^+) \subseteq H$, and (a) holds. \square

We close this section with the following observation,

Remark 6.10. We get the same constraint in (5) for every choice of $\rho \in \text{aff}(F^+) \setminus \text{aff}(F)$.

Proof. Pick $\rho_1, \rho_2 \in \text{aff}(F^+) \setminus \text{aff}(F)$ and for $j = 1, 2$ denote by (ej) the equation (5) with $\rho = \rho_j$. Suppose that u is a solution to (e1) and let $\bar{w}^\top x = \bar{\tau}$ as defined in (12). By the equivalence between (a) and (c) for (e1) in Proposition 6.9, $F^+ \subseteq \{x \in \mathbb{R}^n : \bar{w}^\top x = \bar{\tau}\}$. By the equivalence between (a) and (c) for (e2), u is a solution to (e2). Therefore, (e1) and (e2) have the same set of solutions, and hence, (e1) and (e2) are related by scaling. As the right-hand-sides of (e1) and (e2) both equal 1, the result follows. \square

³Given $S \subseteq \mathbb{R}^n$ and $\rho \in \mathbb{R}^n$, $S - \rho := \{s - \rho : s \in S\}$.

6.5 The proof of Theorem 2.2

We require a number of preliminaries for the proof.

Proposition 6.11. *For w admissible, let F denote the set of optimal solutions of $(P : M, b, w)$ and let F^+ be any down-face of F . If y is an optimal solution of $(D : M, b, w)$ then setting $u_i := y_i$ for all $i \in I_{M,b}(F) \setminus I_{M,b}(F^+)$ satisfies the (w, F, F^+) -tilt constraint.*

Proof. Throughout the proof, we write I for $I_{M,b}$. For every $i \in I(F) \setminus I(F^+)$ set $u_i = y_i$. By Complementary Slackness (10), we have $y_i = 0$ for all $i \notin I(F)$. Since y is feasible for $(D : M, b, w)$, $w = M^\top y$. Therefore,

$$w = \sum_{i \in I(F)} \text{row}_i(M)^\top y_i = \sum_{i \in I(F) \setminus I(F^+)} \text{row}_i(M)^\top u_i + \sum_{i \in I(F^+)} \text{row}_i(M)^\top y_i. \quad (\star)$$

Let \bar{w} be as defined in (12). Then (\star) implies that $\bar{w} = \sum_{i \in I(F^+)} \text{row}_i(M)^\top y_i$. The result now follows by the equivalence of (b) and (c) of Proposition 6.9. \square

Consider $M \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and let $\mathbb{L} \subset \mathbb{R}$. We say that $w \in \mathbb{L}^n$ is \mathbb{L} -bad with respect to $Mx \leq b$, if (i) w is admissible for $(P : M, b, w)$ and (ii) $(D : M, b, w)$ has no optimal solution in \mathbb{L}^m . By definition, $Mx \leq b$ is TD in \mathbb{L} if and only if there is no $w \in \mathbb{L}^n$ that is \mathbb{L} -bad. A vector $w \in \mathbb{L}^n$ is \mathbb{L} -extremal with respect to $Mx \leq b$, if it is \mathbb{L} -bad and among all \mathbb{L} -bad vectors it maximizes the dimension of the set of optimal solutions of $(P : M, b, w)$. We have the following relation between extremal weights and the tilt-constraints.

Proposition 6.12. *Let $M \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and let \mathbb{L} be a heavy set. Suppose that w is \mathbb{L} -extremal with respect to $Mx \leq b$. Let F denote the set of optimal solutions of $(P : M, b, w)$ and let F^+ be a down-face of F . Then the (w, F, F^+) -tilt constraint has no solution with all variables in \mathbb{L} .*

Proof. We write I for $I_{M,b}$ and let $S := \{\text{row}_i(M) : i \in I(F^+)\}$.

Claim 1. *The set S forms a \mathbb{L} -GSS.*

Proof of Claim. Let $\Omega \in \mathbb{L}^n$ be an arbitrary vector in the conic hull of S . Denote by F^Ω the optimal face of $(P : M, b, \Omega)$. By Proposition 6.6, $F^+ \subseteq F^\Omega$. Since $\dim(F^\Omega) \geq \dim(F^+) > \dim(F)$ and since w is extremal, $(D : M, b, \Omega)$ has an optimal solution $z \in \mathbb{L}^m$. By (10), $z_i = 0$ for all $i \notin I(F^\Omega)$. As $F^+ \subseteq F^\Omega$, $I(F^+) \supseteq I(F^\Omega)$, hence, $z_i = 0$ for all $i \notin I(F^+)$. Hence,

$$\Omega = M^\top z = \sum_{i \in I(F^+)} \text{row}_i(M) z_i,$$

and Ω is a conic combination of vectors in S with coefficients in \mathbb{L} . Since Ω was arbitrary, S forms an \mathbb{L} -GSC. The result then follows from Proposition 6.3. \diamond

Suppose for a contradiction that the (w, F, F^+) -tilt constraint has a solution with $u_i \in \mathbb{L}$ for all $i \in I(F) \setminus I(F^+)$. Define,

$$\bar{w}^\top = w^\top - \sum_{i \in I(F) \setminus I(F^+)} u_i \text{row}_i(M). \quad (13)$$

By the equivalence between (b) and (c) of Proposition 6.9, \bar{w} is in the span of S . It then follows from the claim that there exists a vector z where $z_i \in \mathbb{L}$ for all $i \in I(F^+)$ for which,

$$\bar{w}^\top = \sum_{i \in I(F^+)} z_i \text{row}_i(M). \quad (14)$$

Define, \bar{y} where for every $i \in [m]$,

$$\bar{y}_i := \begin{cases} z_i & \text{if } i \in I(F^+) \\ u_i & \text{if } i \in I(F) \setminus I(F^+) \\ 0 & \text{otherwise.} \end{cases}$$

By (13) and (14) we have $M^\top \bar{y} = w$. Since in addition, $\bar{y}_i = 0$ for all $i \notin I(F)$, Proposition 6.8 implies that \bar{y} is in the affine hull of optimal solutions to $(D : M, b, w)$. By construction, $\bar{y} \in \mathbb{L}^m$ and Theorem 6.1 implies that there is an optimal solution to $(D : M, b, w)$ in \mathbb{L}^m . Hence, w is not \mathbb{L} -bad, a contradiction, as w is \mathbb{L} -extremal. \square

We are now ready for the main proof in this section.

Proof of Theorem 2.2. Throughout this proof we write I for $I_{M,b}$. We first assume that $Mx \leq b$ is TD in \mathbb{L} and will show that both (i) and (ii) hold. Consider first (i). Let $\Omega \in \mathbb{L}^n$ be a conic combination of the rows of M^\ominus . It follows from Proposition 6.6 that every feasible solution of $(P : M, b, \Omega)$ is an optimal solution. Pick x in the relative interior of $\{x \in \mathbb{R}^n : Mx \leq b\}$. Since $Mx \leq b$ is TD in \mathbb{L} there exists an optimal solution $y \in \mathbb{L}^m$ of $(D : M, b, \Omega)$. By (10), $y_i = 0$ for all i that does not correspond to a row of M^\ominus . Since $y \geq \mathbf{0}$ and $\Omega = M^\top y$, Ω is a conic combination of M^\ominus with coefficients in \mathbb{L} . As Ω was arbitrary the rows of M^\ominus forms a GSC in \mathbb{L} . Consider (ii). Since $Mx \leq b$ is TD in \mathbb{L} , there exists an optimal solution $y \in \mathbb{L}^m$ of $(D : M, b, w)$. Then by Proposition 6.11, $u_i = y_i$ for all $i \in I(F) \setminus I(F^+)$ satisfies the (w, F, F^+) -tilt constraint.

Assume now that (i) and (ii) hold and suppose for a contradiction that $Mx \leq b$ is not TD in \mathbb{L} . Then there exists w that is \mathbb{L} -extremal. Denote by F the optimal face for $(P : M, b, w)$. Consider first the case where $F = \{x \in \mathbb{R}^n : Mx \leq b\}$. Proposition 6.6 implies that w is a conic combinations of the rows of M^\ominus . Since the rows of M^\ominus form an \mathbb{L} -GSC, there exists $y \in \mathbb{L}_+^m$ with $w = M^\top y$ where $y_i = 0$ for every i that does not correspond to a row of M^\ominus . It follows from (10) that y is an optimal

solution of $(D : M, b, w)$, a contradiction as w is \mathbb{L} -bad. Thus, we may assume F is a proper face of $\{x \in \mathbb{R}^n : Mx \leq b\}$. Therefore, there exists a down-face F^+ of F . By (ii) we have a solution of the (w, F, F^+) -tilt constraint with all variables in \mathbb{L} . However, this contradicts Proposition 6.12. \square

We leave it as an exercise to check that the argument in the proof of Theorem 2.2 also shows that the condition (ii) in that theorem can be replaced by the following condition: “For every admissible $w \in \mathbb{L}^n$, denote by F the set of optimal solutions of $(P : M, b, w)$. For some down-face F^+ of F , the (w, F, F^+) -tilt constraint has a solution with variables in \mathbb{L} .”

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