Pebbling Graphs of Fixed Diameter

Luke Postle*

January 16, 2013

Abstract

Given a configuration of indistinguishable pebbles on the vertices of a connected graph G on n vertices, a *pebbling move* is defined as the removal of two pebbles from some vertex, and the placement of one pebble on an adjacent vertex. The *m-pebbling number* of a graph G, $\pi_m(G)$, is the smallest integer k such that for each vertex v and each configuration of k pebbles on G there is a sequence of pebbling moves that places at least m pebbles on v. When $m = 1$, it is simply called the pebbling number of a graph.

We prove that if G is a graph of diameter d and $k, m \geq 1$ are integers, then $\pi_m(G) \leq f(k)n + 2^{k+d}m + (2^k(2^d - 1) - f(k))dom_k(G)$, where $dom_k(G)$ denotes the size of the smallest distance k dominating set, that is the smallest subset of vertices such that every vertex is at most distance k from the set, and, $f(k) = (2^k - 1)/k$. This generalizes the work of Chan and Godbole [4] who proved this formula for $k = m = 1$. As a corollary, we prove that $\pi_m(G) \leq f(\lceil d/2 \rceil)n + \mathcal{O}(m + \sqrt{n \ln n})$. Furthermore, we prove that if d is odd, then $\pi_m(G) \leq f(\lceil d/2 \rceil)n + \mathcal{O}(m)$, which in the case of $m = 1$ answers for odd d, up to a constant additive factor, a question of Bukh [3] about the best possible bound on the pebbling number of a graph with respect to its diameter.

1 Introduction

A recent development in graph theory, suggested by Lagarias and Saks (via a private communication to Chung), is called *pebbling*. Pebbling was first introduced into the literature by Chung who computed the pebbling number of Cartesian products to give a combinatorial proof of the following number-theoretic statement of Kleitman and Lemke.

Theorem 1. *[5][14] Let $\mathbb{Z}_n$ be the cyclic group on n elements and let $|g|$ denote the order of a group element $g \in \mathbb{Z}_n$. For every sequence $g_1, g_2, \ldots, g_n$ of (not necessarily distinct) elements of $\mathbb{Z}_n$, there exists a zero-sum subsequence $(g_k)_{k \in K}$, such that $\sum_{k \in K} \frac{1}{|g_k|} \leq 1$.*

Chung developed the pebbling game to give a more natural proof of this theorem. Theorems of this type are an important area of study in number theory as they generalize zero-sum theorems such as the Erdős–Ginzburg–Ziv [9] theorem. Geroldinger [10] and then Elledge and Hurlbert [8] generalized Theorem 1 to Abelian groups. The latter work used graph pebbling to do so and also generalized the goal of zero-sum to a sum living in a given normal subgroup. Indeed, over the last twenty years, pebbling has developed into its own subfield [12, 13], with over sixty papers.

A *pebbling configuration* on a graph is a distribution of indistinguishable objects called *pebbles* on vertices of that graph. That is, a pebbling configuration p on a graph G is a function $p\colon V(G) \mapsto \mathbb{N} \cup \{0\}$, where $p(v)$ is the number of pebbles on v in p. A *pebbling move* is defined as the removal of two pebbles from some vertex and the subsequent placement

*Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia, 30322, luke@mathcs.emory.edu; this research was completed during a previous affiliation with the School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332

of one pebble on an adjacent vertex. Hence, a pebbling move transforms one pebbling configuration to a different pebbling configuration.

We say the ordered pair (G, r) is a *rooted graph* if G is a graph and $r \in V(G)$. We say a pebbling configuration p is *m-potent* for a rooted graph (G, r), if there exists a pebbling configuration p' obtained by a sequence of pebbling moves from p such that r has at least m pebbles in p'. We say a pebbling configuration p is *m-impotent* if there does not exist such a pebbling configuration. As in [5], we define the *m-pebbling number*, $\pi_m(G)$, to be the least integer k such that, for any choice of root $r \in V(G)$ and any initial configuration p of k pebbles, p is m-potent for (G, r). The pebbling number refers to the 1-pebbling number of a graph. Notice that a trivial lower bound for $\pi_1(G)$ is $|V(G)|$: Choose $r \in V(G)$ and let $p(r) = 0$ and $p(v) = 1$ for all $v \neq r$. Then p is 1-impotent for (G, r).

The diameter of a graph can also yield lower bounds on its pebbling number. For instance, because the pebbling number of a path on d vertices is 2^{d-1}, then the pebbling number of graphs with diameter d must be a least 2^d. It is then a natural question to ask whether restricting the diameter can give an upper bound on the pebbling number. To that end, define $\pi_m(n, d)$ to be the maximum m-pebbling number of a diameter d graph on n vertices.

For graphs of diameter two much is known. Pachter, Snevily and Voxman in [15] proved that $\pi_1(n, 2) = n + 1$. Clarke, Hochberg and Hurlbert in [6] classified graphs of diameter two whose pebbling number is $n + 1$. Curtis, et al. [7] proved that $\pi_m(n, 2) \leq n + 7m - 6$ and conjectured that $\pi_m(n, 2) \leq n + 4m - 3$, which was recently proved by Herscovici, et al. [11].

As for graphs of larger diameter, more recent results have provided insight for graphs of diameter three and four. Bukh [3] proved that $\pi_1(n, 3) = 3n/2 + \mathcal{O}(1)$. Postle, Streib and Yerger [17] proved an exact bound for $\pi_1(n, 3)$, namely that $\pi_1(n, 3) = \lfloor 3n/2 \rfloor + 2$. They also gave shorter proofs of the aforementioned diameter two results using their new techniques. Furthermore, they proved that $\pi_1(n, 4) = 3n/2 + \Theta(1)$. As for general diameter results, Bukh proved that $\pi_1(n, d) \leq (2^{\lceil d/2 \rceil} - 1)n + \mathcal{O}(\sqrt{n})$. The best known lower bound is $\pi_m(n, d) \geq f(\lceil d/2 \rceil)(n - (d + 1)) + 2^d m$, which comes from a generalization of the example in [3].

In section two, we define branches, which were previously introduced in [17], and prove several fundamental results about them. In section three, we use these results to bound pebbling numbers in terms of domination numbers.

Let G be a graph. We say that S is a *k-dominating set* if every vertex in G is either in S or is at most distance k from some vertex in S. The *k-domination number*, denoted by $dom_k(G)$, is the size of the smallest k-dominating set in G. When $k = 1$, this is the domination number.

We prove the following theorem relating pebbling number and k-domination numbers.

Theorem 2. *Let G be a graph on n vertices of diameter d. For all $k, m \geq 1$, $\pi_m(G) \leq f(k)n + 2^{k+d}(m - 1) + (2^k(2^d - 1) - f(k))dom_k(G)$, where $f(k) = (2^k - 1)/k$.*

This generalizes a result of Chan and Godbole [4] who proved that $\pi_1(G) \leq n + (2^{d+1} - 3)dom(G)$.

As a corollary to this theorem, we obtain a bound on $\pi_m(n, d)$.

Corollary 1. *If G is a graph on n vertices of diameter d, then $\pi_m(G) \leq f(\lceil d/2 \rceil)n + 2^{d+\lceil d/2 \rceil}(m - 1) + (2^{\lceil d/2 \rceil}(2^d - 1) - f(\lceil d/2 \rceil))\sqrt{n \ln n}$.*

Hence, $\pi_m(n, d) \leq f(d/2)n + \mathcal{O}(m + \sqrt{n \ln n})$, which is best possible up to a sublinear asymptotic factor. It is worth comparing this to the recent work of Herscovici, et al. who proved that $\pi_m(n, d) \leq f(d)(n - 1) + 2^d(m - 1) + 1$.

We also prove the following theorem:

Theorem 3. *If d is an odd positive integer, then $\pi_m(n, d) \leq f(\lceil d/2 \rceil)n + \mathcal{O}(m)$.*

Given the lower bounds mentioned above this proves that

Corollary 2. *If d is an odd positive integer, then $\pi_1(n,d) = \theta(f(\lceil d/2 \rceil)n)$.*

2 Branches

Let S be a subset of the vertices. We say that a spanning forest T of G is a *breadth-first search (BFS) spanning forest* of G with *root set* S if, for every vertex $v \in V(G)$, the shortest path from v to S in T is also a shortest path from v to S in G, and every vertex v in S is contained in a different component of T. We will use the standard notions from BFS trees of descendant, child, parent, and ancestor for BFS forests as well. We also let $d(u,v)$ denote the distance between two vertices.

Definition. We say the ordered triple (B,p,w) is a *branch* if (B,w) is a rooted tree and p is a pebbling configuration on $V(B)$. Where p and w are understood, we will say that B is branch. The *depth* of a branch (B,p,w), to be denoted by $d(B)$, is the maximum distance in B from a vertex in B to w.

We also define $p(B)$ to be the number of pebbles in the branch, that is, $\sum_{v \in V(B)} p(v)$.

Definition. Let (B,p,w) be a branch of depth $k>0$. We define the *truncation* of B to be the branch (B',p',w) obtained from B by making pebbling moves to move as many pebbles as possible from all the vertices of depth k to their parents and then deleting the vertices of depth k. If $i \leq k$, we define the *i-truncation* of B, to be denoted by $B^{(i)}$, as the branch obtained from B by successively truncating it i times.

Definition. We define the *potency* of branch (B,p,w), to be denoted as $\overline{p}(B)$, as $p(B^{(d(B))})$. Similarly we define the *capacity*, denoted by $c(B)$, as $\lfloor \overline{p}(B)/2 \rfloor$.

If u is a vertex in a branch B we will let $B[u]$ denote the subbranch (B_u,p,u) of B, where B_u is the subtree of B induced by u and all of its descendants. A branch (B,p,w) is *irreducible* if for all vertices $v \in B$, where $v \neq w$, $B[v]$ has nonzero capacity. If B is not irreducible, we will say that B is *reducible.* If B is reducible and $u \in V(B)$ such that $u \neq w$ and $B[u]$ has zero capacity, then B may be decomposed into two branches $B \setminus B[u]$ and $B[u]$. Continuing this process, we may decompose B into irreducible branches; moreover this decomposition is unique as the roots of these branches must exactly correspond to the vertices $u \in V(B)$ such that $u = w$ or $B[u]$ has zero capacity. We refer to this decomposition as the *irreducible decomposition* of B.

Definition. We define the function $F(k)$ to be the supremum of $\frac{p(B)}{|V(B)|}$ over all branches (B,p,w) of zero capacity and depth at most $k-1$. For all $k \geq 1$, we define the *k-excess* of a branch B, denoted $X_k(B)$, to be $p(B) - F(k)|V(B)|$.

Proposition 1. *There are only finitely many irreducible branches of depth at most d and potency l.*

Proof. We proceed by induction on d. If $d=0$, then such a branch is simply a vertex and the number of pebbles on that vertex is l. So we may assume that $d \geq 1$. Now in an irreducible branch of potency l, the root w_0 has at most l children. The subbranches induced by the children of w_0 have depth at most $d-1$. As the number of possible such subbranches is finite by induction and the range of possible values for $p(w_0)$ is also finite, there are at most a finite number of possible branches of depth at most d and potency l. □

Lemma 1. *$F(k)$ is equal to the maximum of $\frac{p(B)}{|V(B)|}$ over all irreducible branches of zero capacity and depth at most $k-1$.*

Proof. By Proposition 1, the supremum over irreducible branches is indeed a maximum. Let (B, p, w) be a branch of zero capacity and depth at most $k-1$. Consider the irreducible decomposition of B into irreducible branches $B_1, \ldots, B_t$, which have zero capacity and depth at most $k-1$. As $p(B) = \sum_{i=1}^{t} p(B_i)$, $\frac{p(B)}{|V(B)|} = \sum_{i=1}^{t} \frac{p(B_i)}{|V(B_i)|} \frac{|V(B_i)|}{|V(B)|}$. If we let c denote the maximum of $\frac{p(B)}{|V(B)|}$ over irreducible branches of zero capacity and depth at most $k-1$, then this is at most $c \sum_{i=1}^{t} \frac{|V(B_i)|}{|V(B)|}$. Since $|V(B)| = \sum_{i=1}^{t} |V(B_i)|$, this is at most c as desired. □

Lemma 2. *For all $d \leq k$, the supremum of $X_k(B)$ over all branches of depth at most d and potency l is equal to the maximum of $X_k(B)$ over all irreducible branches of depth at most d and potency l.*

Proof. Let c be the maximum of $X_k(B)$ over all irreducible branches of depth at most d and potency l. Such a maximum exists by Proposition 1. It suffices to prove that if (B, p, w_0) is a branch of depth at most d and potency l, then $X_k(B) \leq c$. We proceed by induction on d and then induction on $|V(B)|$. If B is irreducible, this follows from the definition of c. If $d = 0$, then B is simply a vertex and so irreducible and the lemma follows.

So we may assume that $d \geq 1$ and B is not irreducible. Then there exists $u \in V(B)\backslash\{w_0\}$ such that $B[u]$ has capacity zero. Yet, $B' = B \setminus B[u]$ is a branch with depth at most d, potency l, and a smaller number of vertices. In addition, $X_k(B') = X_k(B) - X_k(B[u])$. Since $B[u]$ has depth at most $d-1$ which is at most $k-1$, $X_k(B[u]) \leq 0$ by the definition of $F(k)$. Hence, $X_k(B) \leq X_k(B') \leq c$ as desired. □

Lemma 3. *Let d, l be non-negative integers and let k be an integer such that $k \geq d$. Suppose that (B, p, w_0) is an irreducible branch of depth at most d and potency l such that B has maximum k-excess and, subject to that condition, has a minimum number of vertices. It follows that $B = w_0 w_1 \ldots w_{d(B)}$ is a path such that $p(w_i) = 0$ for all i, $0 \leq i < d(B)$.*

Proof. We claim that if $u \in V(B)$ is not a leaf, then $p(u) = 0$. Suppose not. Define a new pebbling configuration p' on $V(B)$ as follows. Let v be a child of u. Let $p'(u) = p(u) - 1$, $p'(v) = p(v) + 2$ and $p'(z) = p(z)$ for all other vertices $z \neq u, v$. The branch (B, p', w_0) has depth at most d and potency l. However, $p'(B) > p(B)$ and thus (B, p', w_0) has a larger k-excess, a contradiction.

Finally we claim that all vertices in B have at most one child. Suppose not. Let v be a vertex with at least two children but such that every descendant of v has at most one child. Let u_1, u_2 be two children of v. Let t_1 be the descendant of u_1 that is a leaf and t_2 be the descendant of u_2 that is a leaf. We may assume without loss of generality that $d(v, t_1) \geq d(v, t_2)$. Define a new pebbling configuration p' on $V(B)$ as follows. Let q be the largest integer such that $p(t_2) \geq q2^{d(v,t_2)}$. Let $p'(t_2) = p(t_2) - q2^{d(v,t_2)}$, $p'(t_1) = p(t_1) + q2^{d(v,t_1)}$ and $p'(z) = p(z)$ for all other vertices $z \neq t_1, t_2$. The branch (B, p', w_0) has depth at most d and potency l. Moreover, $p'(B) \geq p(B)$ and thus must also have maximum excess. However, the induced subbranch $B[u_2]$ of (B, p', w_0) has capacity zero since $p'(t_2) \leq 2^{d(v,t_2)} - 1$ and $p'(x) = 0$ for all other $x \in B[u_2]$. Hence (B, p', w_0) is not irreducible, a contradiction. □

Corollary 3. *For all $k \geq 1$, $F(k) = f(k)$.*

Proof. By Lemma 1, the maximum k-excess among branches of depth at most $k-1$ and zero capacity is attained at some irreducible branch (B, p, w_0). By Lemma 3, we may assume that B is a path $w_0 w_1 \ldots w_{d(B)}$ and $p(w_i) = 0$ for all i, $0 \leq i < d(B)$. Now p is 2-impotent for (B, w_0) if and only if $p(w_{d(B)}) \leq 2^{d(B)+1} - 1$. The maximum k-excess given depth $d(B)$ would thus be obtained when $p(w_{k-1}) = 2^{d(B)} - 1$ and $X_k(B) = p(B) - F(k)|V(B)| = 2^{d(B)} - 1 - F(k)(d(B) + 1)$. However as B has maximum k-excess, $X_k(B) = 0$. Thus, $F(k) = \frac{2^{d(B)+1}-1}{d(B)+1}$. Certainly this is maximized when $d(B)$ is maximized, that is when $d(B) = k - 1$. Thus $F(k) = \frac{2^k-1}{k} = f(k)$ as desired. □

Corollary 4. *The maximum k-excess over branches of depth at most k and potency l is $2^k l - f(k)$. Hence, if B is branch of depth at most k, then $X_k(B) \leq 2^k \overline{p}(B) - f(k)$.*

Proof. By Lemma 1, the maximum k-excess among branches of depth at most $k-1$ and potency l is attained at some irreducible branch (B, p, w_0). By Lemma 3, we may assume that B is a path $w_0 w_1 \dots w_{d(B)}$ and $p(w_i) = 0$ for all i, $0 \leq i < d(B)$. As B has potency l, $p(w_{d(B)}) \leq 2^{d(B)}(l+1)-1$. The maximum k-excess for a branch of depth $d(B)$ would thus be obtained when $p(w_{k-1}) = 2^{d(B)}(l+1) - 1$. Hence, $X_k(B) = p(B) - f(k)|V(B)| = 2^{d(B)}(l+1) - 1 - \frac{(2^k-1)}{k}(d(B)+1)$. It is not hard to see that the maximum k-excess is obtained when the depth is maximized, that is when $d(B) = k-1$ and hence $X_k(B) = 2^k(l+1) - F(k)k$. Hence, $X_k(B) = 2^k(l+1) - 1 - \frac{(2^k-1)}{k}(k+1) = 2^k l - f(k)$ as desired.

If B is a branch of depth at most k, its potency is $\overline{p}(B)$. Thus, $X_k B$ is a most the maximum k-excess over branches of depth at most k and potency $\overline{p}(B)$ which is $2^k \overline{p}(B) - f(k)$. □

3 Dominating Sets

The following theorem was proved by Arnautov and independently by Payan.

Theorem 4. *[2][16] If G is a graph with minimum degree $\delta(G)$, then $dom(G) \leq n(1 + \ln(\delta(G)+1))/(\delta(G)+1)$.*

The following are two useful corollaries of this theorem. The first was proved asymptotically by Al-Yakoob and Tuza [1] with a slightly better bound.

Corollary 5. *If G is a graph of diameter at most two on $n \geq 3$ vertices, then $dom(G) \leq \sqrt{n \ln n}$.*

Proof. Let $v \in V(G)$. Notice that $N(v)$ is a dominating set in G as G is diameter two. Hence, $dom(G) \leq \delta(G)$. Thus if $\delta(G) \leq \sqrt{n \ln n}$, Corollary 5 holds. So suppose $\delta(G) \geq \sqrt{n \ln n}$. By Theorem 4, $dom(G) \leq n(1 + \ln(\delta(G)+1))/(\delta(G)+1) \leq n(1 + \ln(n)/2 + \ln\ln(n)/2)/\sqrt{n \ln n}$. However this is at most $n \ln(n)/\sqrt{n \ln n} = \sqrt{n \ln n}$ since $n \geq 3$, and Corollary 5 holds. □

Corollary 6. *If G is a graph of diameter d on $n \geq 3$ vertices, then $dom_{\lceil d/2 \rceil}(G) \leq \sqrt{n \ln n}$.*

Proof. Apply Corollary 5 to the graph G' where $V(G') = V(G)$ and there is an edge between two vertices x and y if and only if $d(x,y) \leq \lceil d/2 \rceil$. □

Now we are prepared to prove the main theorem.

Proof of Theorem 2. Let $r \in V(G)$ and let p be a pebbling configuration that is m-impotent for (G, r). Let S be a smallest k-dominating set in G. Let T be a BFS spanning forest with root set $S \cup r$. For every $s \in S \cup r$, let C_s denote the component of T containing s. Notice that (C_s, p, s) is a branch of depth at most k.

Note that $\sum_{v \in V(G)} p(v) = \sum_{s \in S \cup r} p(C_s) = f(k)n + \sum_{s \in S \cup r} X_k(C_s)$. Moreover, if $\overline{p}(C_s) \geq q2^d$ then C_s can send q pebbles to r using only the pebbles in C_s. For all $s \in S$, let q_s be the largest integer such that $\overline{p}(C_s) \geq q_s 2^d$. Let $q_r = \overline{p}C_r$. Since p is m-impotent, it follows that $\sum_{s \in S} q_s + q_r \leq m - 1$. Thus, $\sum_{s \in S \cup r} \overline{p}(C_s) \leq \sum_{s \in S}(2^d q_s + 2^d - 1) + q_r \leq 2^d(m-1) + (2^d-1)(dom_k(G) - q_r)$. By Corollary 4, $X_k(C_s) \leq 2^k \overline{p}(C_s) - f(k)$ for all $s \in S \cup r$. Hence, $\sum_{v \in V(G)} p(v) \leq f(k)n + 2^{k+d}(m-1) + (2^k(2^d-1) - f(k))dom_k(G) - 2^k(2^d-1)q_r - f(k)$. As $f(k) \geq 1$ since $k \geq 1$ and $q_r \geq 0$, this at most one less than the formula desired in Theorem 2. Since $\pi_m(G) - 1$ is equal to the maximum number of pebbles over all m-impotent configurations, Theorem 2 holds. □

Proof of Corollary 1. Apply Theorem 2 with $k = \lceil d/2 \rceil$. By Corollary 6, $dom_{\lceil d/2 \rceil}(G) \le \sqrt{n \ln n}$. □

Finally, we improve on our bound for odd d to obtain a bound that is best possible up to a constant additive factor.

Theorem 5. *If G is a graph on n vertices of odd diameter d, then $\pi_m(G) \le f(\lceil d/2 \rceil)n + 2^d 2^{\lceil d/2 \rceil}(m-1) + 8^{d+4/3}$.*

Proof. If $d = 1$, then $dom(G) = 1$ and Theorem 5 follows by Theorem 2 with $k = 1$. So suppose $d \ge 3$. Consider $G^{\lfloor d/2 \rfloor}$, the graph with vertex set $V(G)$ where for all $x, y \in V(G)$, x is adjacent to y if and only if $d(x, y) \le \lfloor d/2 \rfloor$. Let $\alpha = \delta(G^{\lfloor d/2 \rfloor}) + 1$. Note that the $\lfloor d/2 \rfloor$-neighborhood of any vertex is a $\lceil d/2 \rceil$-dominating set in G. In other words, $dom_{\lceil d/2 \rceil}(G) \le \alpha - 1$. Yet we also know that any dominating set in $G^{\lfloor d/2 \rfloor}$ is a $\lfloor d/2 \rfloor$-dominating set in G. Hence by Theorem 4, $dom_{\lfloor d/2 \rfloor}(G) \le n \ln(\alpha)/\alpha$.

Now we condition on α. If $\alpha \le 16(2^d 2^{\lfloor d/2 \rfloor})$, apply Theorem 2 with $k = \lceil d/2 \rceil$. We obtain the following bound as desired: $\pi_m(G) \le f(\lceil d/2 \rceil)n + 2^d 2^{\lceil d/2 \rceil}(m-1) + 8^{d+4/3}$.

If $\alpha \ge 16(2^d 2^{\lfloor d/2 \rfloor})$, apply Theorem 2 with $k = \lfloor d/2 \rfloor$. We obtain the following bound: $\pi_m(G) \le f(\lfloor d/2 \rfloor)n + 2^d 2^{\lfloor d/2 \rfloor}(m - 1 + n \ln(\alpha)/\alpha)$. Under these assumptions, $2^d 2^{\lfloor d/2 \rfloor} \ln(\alpha)/\alpha \le (\ln(16) + 3\ln(2)d/2)/16 = \ln(2)(8 + 3d)/32$. Note that for all $d \ge 3$ the difference between $f(\lceil d/2 \rceil)$ and $f(\lfloor d/2 \rfloor)$ is at least $d/6$. Since $d/6 > \ln(2)(8 + 3d)/32$ for all $d \ge 3$, we can merge the two linear terms into one to obtain the following bound: $\pi_m(G) \le f(\lceil d/2 \rceil) + 2^d 2^{\lfloor d/2 \rfloor}(m-1)$. □

References

[1] S. M. Al-Yakoob and Z. Tuza, Domination number of graphs with bounded diameter, JCMCC **40** (2002), 183–191.

[2] V. I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, Prikl Mati Programmirovante **11** (1994), 3–8, 126.

[3] B. Bukh, Maximum pebbling number of graphs of diameter three, J. Graph Theory **52**(2006), 353–357.

[4] M. Chan and A. Godbole, Improved Pebbling Bounds, Discrete Math **208** (2008), 15–23.

[5] F. Chung, Pebbling in hypercubes, SIAM J. Discrete Mathematics **2** (1989), 467–472.

[6] T. A. Clarke, R. A. Hochberg, and G. H. Hurlbert, Pebbling in diameter two graphs and products of paths, J. Graph Theory **25** (1997), 119–128.

[7] D. Curtis, T. Hines, G. Hurlbert, and T. Moyer, Pebbling graphs by their blocks, Integers: Elec. J. Combin. Number. Th., **9** (2009), G2, 411–422.

[8] S. Elledge and G. Hurlbert, An application of graph pebbling to zero-sum sequences in Abelian groups, Integers: Elec. J. Combin. Number Th., **5**(1) (2005), A17.

[9] P. Erdős, A. Ginzburg and A. Ziv, A theorem in additive number theory, Bull. Res. Council Isreal, **10F** (1961), 41–43.

[10] A Geroldinger, On a conjecture of Kleitman and Lemke, J. Number Theory **44** (1993), 60–65.

[11] D. Herscovici, B. Hester and G. Hurlbert, *t*-Pebbling and extensions, to appear in Graphs and Combinatorics.

[12] G. Hurlbert, A survey of graph pebbling, Congr. Numer. **139** (1999), 41–64.

[13] G. Hurlbert, Recent progress in graph pebbling, Graph Theory Notes of New York **XLIX** (2005), 25–37.

[14] D. Kleitman and P. Lemke, An addition theorem on the integers modulo n, J. Number Theory **31** (1989), 335–345.

[15] L .Pachter, H. S. Snevily, and B. Voxman, On pebbling graphs, Proceedings of the Twenty-Sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), vol.107, (1995), 65–80.

[16] C. Payan, Sur le nombre d'absorption d'un graphe simple, Cahiers Centre Etudes Recherche Oper. **17** (1975), no. 2-4, 307–317.

[17] L. Postle, N. Streib and C. Yerger, Pebbling Graphs of Diameter Three and Four, to appear in J. Graph Theory.