

**Question 1**

Indicate, for each of the following statements, whether it is true or false. Write “True” or “False” next to each item. No explanation is necessary.

- Let \( f \in C^0(\mathbb{R}^n) \). If, for all \( \alpha \in \mathbb{R} \), there exists \( \delta > 0 \) such that \( \{ x \in \mathbb{R}^n : f(x) \leq \alpha \} \subseteq B_\delta(0) \), then \( f \) is coercive.

- Let \( f : \mathbb{R}^n \to \mathbb{R} \) be defined by \( q(x) = x^T A x + b^T x \). If \( A \) is positive semidefinite, then \( q \) has a unique global minimizer.

- Let \( f : \mathbb{R}^n \to \mathbb{R} \) be defined by \( f(x) = ||Ax - b||_2^2 \). If \( A \) is not positive definite, then \( f \) has no global minimizer.

- Let \( f : \mathbb{R}^n \to \mathbb{R} \), and let \( p \in \mathbb{R}^n \) be a descent direction at \( x \in \mathbb{R}^n \). If \( f(x + \alpha p) \leq f(x) + \sigma \alpha \nabla f(x)^T p \), then \( \alpha \) satisfies the sufficient decrease condition (where \( 0 < \sigma < \frac{1}{2} \)).

- Let \( f \) be a strictly convex quadratic function. Let Newton’s method be initialized with \( x^0 = 0 \). Then, \( x^1 \) is the unique global minimizer for \( f \).

- Let \( f \) be a quadratic function with a unique global minimizer \( x^* \in \mathbb{R}^n \). If the trust region method is initialized with a trust region radius \( \delta^0 > 0 \) at a point \( x^0 \in B_{\delta^0}(x^*) \), then \( x^1 = x^* \).

**Question 2**

Consider a function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \frac{1}{3} x^3 - 2x \).

(i) Determine all values of \( x^k \) such that \( x^{k+1} \) is well defined with Newton’s method. Among those, find one that is a local minimizer.

(ii) Determine a value \( r > 0 \) such that if \( x^0 \in B_r(x^*) \), then Newton’s method converges quadratically to \( x^* \).

(iii) Give a counter-example to the following (wrong) claim: With Newton’s method, for all \( x^k \in \mathbb{R} \), \( ||x^{k+1} - x^*||_2 < ||x^k - x^*||_2 \).

**Question 3**

Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix that is not positive semidefinite, and let \( b \in \mathbb{R}^n \) be a vector that is not orthogonal to any eigenvector of \( A \).

(i) Prove that there exists \( \beta \in \mathbb{R} \) such that \( \min\{x^T(A + \beta I)x + b^T x\} \) has an optimal solution \( \hat{x} \) and such that \( ||\hat{x}||_2 = 1 \).

(ii) Prove that \( \hat{x} \) is a global minimizer to \( \min\{x^T A x + b^T x : ||x||_2 = 1\} \).