CO 367 Fall 2018: Homework 3

Due: December 3rd, 1:30pm – extended deadline: December 7th, 1:30pm

Instructions  For every nontrivial step you perform, you must justify why the step is valid and what assumptions it exploits. In other words, you do not need to justify basic algebraic operations (rearranging or distributing terms, multiplying both sides of an equation by a constant, etc.), but you do need to explain all steps that exploit hypotheses and assumptions (positive semidefiniteness of a matrix, continuity or convexity of a function, taking a limit that must exist, etc.). If you exploit a result seen in class, or an elementary theorem, clearly state which one.

Question 1  [4 marks] Find a globally optimal solution to the following problem:

\[
\begin{align*}
\min_{x \in \mathbb{R}^3} & \quad c^T x + x^T A x \\
\text{s.t.} & \quad w^T x \geq 4 \\
& \quad x \geq 0,
\end{align*}
\]

where \(c = [1 \ 1 \ 1]^T\), \(w = [1 \ 2 \ 1]^T\) and

\[
A = \begin{bmatrix}
3 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}.
\]

Justify why this solution is a global minimizer.

Question 2  [3 marks] Let \(A \in \mathbb{R}^{m \times n}\) with \(\text{rank}(A) = m\), let \(t \in \mathbb{R}^n\), and let \(b \in \mathbb{R}^m\). Show that we can solve

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad ||x - t||_2 \\
\text{s.t.} & \quad Ax = b,
\end{align*}
\]

by just solving a system of linear equations. Justify why your approach yields a global minimizer.

Question 2  [3 marks] Let \(Q, R \in \mathbb{R}^{p \times p}\) be two symmetric, positive definite matrices, and let \(u, v \in \mathbb{R}^p\). Show that the following problem

\[
\begin{align*}
\min_{y, z \in \mathbb{R}^p} & \quad ||y - z||_2 \\
\text{s.t.} & \quad (y - u)^T Q (y - u) \leq 1 \\
& \quad (z - v)^T R (z - v) \leq 1
\end{align*}
\]

can be formulated as a conic optimization problem, i.e., a problem of the form

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in K_1 \times \cdots \times K_m,
\end{align*}
\]

for some \(A, b, c\) and where \(K_i\) is one of \(\mathbb{R}^{k_i}_+, C^{k_i+1}_2, S^{k_i}_+\) for all \(i = 1, \ldots, m\). You can leave \(Ax = b\) in linear constraint notation (no need to construct the matrix \(A\) explicitly), but the vector of all variables \((x)\) must belong to a Cartesian product \(K_1 \times \cdots \times K_m\) of closed convex cones \(\mathbb{R}^{k_i}_+, C^{k_i+1}_2,\) or \(S^{k_i}_+\).