CO 367 Fall 2018: Homework 3

Due: December 3rd, 1:30pm – extended deadline: December 7th, 1:30pm

Instructions For every nontrivial step you perform, you must justify why the step is valid and what assumptions it exploits. If you exploit a result seen in class, or an elementary theorem, clearly state which one.

**Question 1** [4 marks] Find a globally optimal solution to the following problem:

\[ \min_{x \in \mathbb{R}^3} \quad c^T x + x^T Ax \]
\[ \text{s.t.} \quad w^T x \geq 4 \]
\[ x \geq 0, \]

where \( c = [1 \ 1 \ 1]^T \), \( w = [1 \ 2 \ 1]^T \) and
\[ A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \]

Justify why this solution is a global minimizer.

*Hint 1:* First, consider the relaxation of this problem obtained by dropping the \( x \geq 0 \) constraints. Then, argue that a globally optimal solution for the resulting relaxed problem is globally optimal for the original problem.

*Hint 2:* You can assume that \( A \) is positive definite. One bonus mark for computing its eigenvalues.

**Question 2** [3 marks] Let \( A \in \mathbb{R}^{m \times n} \) with \( \text{rank}(A) = m \), let \( t \in \mathbb{R}^n \), and let \( b \in \mathbb{R}^m \). Show that we can solve

\[ \min_{x \in \mathbb{R}^n} \quad ||x - t||_2 \]
\[ \text{s.t.} \quad Ax = b, \]

by just solving a system of linear equations. Justify why your approach yields a global minimizer.

**Question 2** [3 marks] Let \( Q, R \in \mathbb{R}^{p \times p} \) be two symmetric, positive definite matrices, and let \( u, v \in \mathbb{R}^p \). Show that the following problem

\[ \min_{y, z \in \mathbb{R}^p} \quad ||y - z||_2 \]
\[ \text{s.t.} \quad (y - u)^T Q (y - u) \leq 1 \]
\[ (z - v)^T R (z - v) \leq 1 \]

can be formulated as a conic optimization problem, i.e., a problem of the form

\[ \min_{x \in \mathbb{R}^n} \quad c^T x \]
\[ \text{s.t.} \quad Ax = b \]
\[ x \in K_1 \times \cdots \times K_m, \]

for some \( A, b, c \) and where \( K_i \) is one of \( \mathbb{R}^{k_i}_+, C_2^{k_i+1}, S_+^{k_i} \) for all \( i = 1, \ldots, m \). You can leave \( Ax = b \) in linear constraint notation (no need to construct the matrix \( A \) explicitly), but the vector of all variables \( x \) must belong to a Cartesian product \( K_1 \times \cdots \times K_m \) of closed convex cones \( \mathbb{R}^{k_i}_+, C_2^{k_i+1}, \) or \( S_+^{k_i} \).