The Theorem of Pappus and Commutativity of Multiplication

Leroy J Dickey *

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Abstract

The purpose of this note is to present a proof of the Theorem of Pappus that reveals the role of commutativity of multiplication. This proof, my current favourite, shows that the Pappus Configuration "closes" if and only if two numbers a and b commute.

1 Introduction

The proof given here is easy to follow and involve algebraic expressions that are simpler than are found in any other proof I have seen. This proof is distinguished by the selection of the points used for the frame of reference. The result is that all the algebraic expressions that arise are as simple as possible and at each step along the way, the property of commutativity is not used.

The frame selection may surprise readers who are familiar with proofs which use the convention that the the frame of reference is chosen from the set of "given" points, and then the co-ordinates of the remaining point and lines are determined. In this context, following this convention would mean that the points of the frame would be chosen from this set of 6 points: A_1, B_1, C_1, A_2, B_2, and C_2. In this proof, we depart from that restriction.

We assume that the plane is coordinatized by a **division ring**. The formal definition is presented below 1.1, but for those who don't want to bother with the formalities of the definition, a good working definition is this: A division ring is almost a field, and the only field property that is not required is that of commutativity of multiplication. Thus a field is a commutative division ring.

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If you are not sure what a field is, you probably know some examples: the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. There are infinitely many more infinite fields. There are also some finite fields. The easiest one is $\mathbb{Z}_2$, the set of integers modulo 2. There are infinitely many more; $\mathbb{Z}_p$, integers mod p for any prime p.

Perhaps the best known example of a non-commutative division ring is the quaternions. They were discovered and promoted by William Rowan Hamilton, and they are still used today.

1.1 Definition of a Division Ring

A division ring is a set R containing at least two elements including 0 and 1 so that for every $a \in R$ and every $b \in R$.

There is a well defined operation "+" so that R with this operation is a commutative group. That is, so that

(1)	+ closure	$(\forall a, b \in R)$	$[a + b \in R]$
(2)	+ identity	$(\exists 0 \in R)(\forall a \in R)$	$[a + 0 = a = 0 + a]$
(3)	+ inverse	$(\forall a \in R)(\exists b \in R)$	$[a + b = 0 = b + a]$
		and we write	$b = -a$
(4)	+ associativity	$(\forall a, b, c \in R)$	$[a + (b + c) = (a + b) + c]$
(5)	+ commutativity	$(\forall a, b \in R)$	$[a + b = b + a]$

Further, there is a second well defined operation "×" or "·" so that

(6)	× closure	$(\forall a, b \in R)$	$[a \cdot b \in R]$
(7)	× identity	$(\exists\ 1 \in R)[1 \neq 0](\forall a \in R)$	$[a \cdot 1 = a = 1 \cdot a]$
(8)	× inverse	$(\forall a \in R \backslash \{0\})(\exists b \in R)$	$[a \cdot b = 1 = b \cdot a]$
		and we write	$b = a^{-1}$
(9)	× associativity	$(\forall a, b, c \in R)$	$[(a \cdot b) \cdot c = a \cdot (b \cdot c)]$
(10)	× distributivity	$(\forall a, b, c \in R)$	$[a \cdot (b + c) = a \cdot b + a \cdot c]$
		$(\forall a, b, c \in R)$	$[(a + b) \cdot c = a \cdot c + b \cdot c]$

2 The context

The context of this proof is the projective plane as coordinatized by homogeneous coordinates coming from a division ring R.

2.1 Points

We write

$$P : (p_1, p_2, p_3)$$

to mean that the point P is represented by the triple (p_1, p_2, p_3) where

$$(p_1, p_2, p_3) \neq (0, 0, 0)$$

and $p_1, p_2, p_3 \in R$.

Also, if $k \neq 0$ and $k \in R$ then for the same point P, we may equally well write

$$P : (p_1 k, p_2 k, p_3 k).$$

2.2 Lines

We write

$$\ell : [l_1, l_2, l_3]$$

to mean that the line ℓ is represented by the triple $[l_1, l_2, l_3]$ and $[l_1, l_2, l_3] \neq [0, 0, 0]$ and $l_1, l_2, l_3 \in R$. If $m \in R$ and $m \neq 0$ we may equally well write

$$\ell : [ml_1, ml_2, ml_3].$$

2.3 Incidence

If $P : (p_1, p_2, p_3)$ is any point, and if $\ell : [l_1, l_2, l_3]$ is any line, and if $k, m \in R$ and $k \neq 0$ and $m \neq 0$, then

$$\begin{aligned} \ell \text{ and } P \text{ are } \textbf{incident} &\iff [l_1, l_2, l_3] \cdot (p_1, p_2, p_3) = 0 \\ &\iff m[l_1, l_2, l_3] \cdot (p_1, p_2, p_3)k = 0. \end{aligned}$$

If ℓ and P are incident, we say that ℓ is on P and dually, that P is on ℓ.

3 The Theorem of Pappus

Let R be a division ring and consider the projective plane coordinatized by that ring, with point and lines denote as above.

Theorem 1. *Let ℓ_1 and ℓ_2 be two distinct lines in the projective plane. Let A_1, B_1 and C_1 be three distinct points on ℓ_1. Let A_2, B_2 and C_2 be three distinct points on ℓ_2. Suppose also that these six points are distinct from the point of intersection $D = \ell_1 \cap \ell_2$. Define three more points A_3, B_3 and A_3 by*

$$A_3 = B_1C_2 \cap B_2C_1, \tag{1}$$
$$B_3 = C_1A_2 \cap C_2A_1, \tag{2}$$
$$C_3 = A_1B_1 \cap A_2B_1. \tag{3}$$

Then A_3, B_3 and C_3 are collinear.

3.1 Notation

We use the notation

$$L : [\ell_1, \ell_2, \ell_3] \quad \Longleftarrow \quad \left\langle \begin{array}{l} P : (p_1, p_2, p_3) \\ Q : (q_1, q_2, q_3) \end{array} \right.$$

to mean that P and Q are distinct points, and they determine the line L by way of these two equations

$$\begin{aligned} [\ell_1, \ell_2, \ell_3] \cdot (p_1, p_2, p_3) &= 0 \\ [\ell_1, \ell_2, \ell_3] \cdot (q_1, q_2, q_3) &= 0 \end{aligned}$$

Throughout the rest of this proof, this we use the convention that lines are written on the left and points are written on the right, even though the logical flow of the argument works by starting with the points and from them we find the coordinates of the line that is on the right.

Similarly, for two distinct lines L and M, the notation

$$\left. \begin{array}{r} L : [\ell_1,\ \ell_2,\ \ell_3] \\ M : [m_1, m_2, m_3] \end{array} \right\rangle \quad \Longrightarrow \quad P : (p_1, p_2, p_3)$$

to mean that the lines L and M determine the point P by the two equations

$$\begin{aligned} [\ell_1, \ell_2, \ell_3] \cdot (p_1, p_2, p_3) &= 0 \\ [m_1, m_2, m_3] \cdot (p_1, p_2, p_3) &= 0. \end{aligned}$$

3.2 The Proof

In this section, we present a proof of Pappus's theorem that can be adapted to prove the converse, namely that if Pappus's theorem holds and our plane is coordinatized by a division ring, then the ring must be commutative.

Proof. Suppose, as stated in the theorem, we are given two lines ℓ_1 and ℓ_2, with A_1, B_1, C_1 on ℓ_1 and A_2, B_2, C_2 on ℓ_2. and these six points are distinct from $\ell_1 \cap \ell_2$. Also, define three more points by

$$\begin{aligned} A_3 &= B_1C_2 \cap B_2A_1 \\ B_3 &= C_1A_2 \cap C_2B_1 \\ C_3 &= A_1B_2 \cap A_2C_1 \end{aligned}$$

Our task is to show that A_3, B_3 and C_3 are collinear. (More precisely, that they are collinear if and only if the division ring is commutative).

Note that because we are working with a division ring, we will not make use of commutativity of multiplication. We begin by choosing the frame of reference to be the four points A_1, A_2, A_3, B_3, in that order, so that their coordinates are given by

1. $A_1 : (1, 0, 0)$.
2. $A_2 : (0, 1, 0)$.
3. $A_3 : (0, 0, 1)$.
4. $B_3 : (1, 1, 1)$.

Figure 1 shows, in part, the strategy of proof. The four points that are the frame of reference that are marked with a diamond and the final point where things come together is C_3

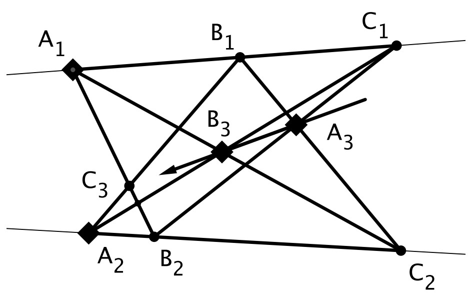

Figure 1: The four points A_1, A_2, A_3 and B_3 are chosen to be the frame of reference.

5. **The line A_1B_3.**
   - From items (1) and (4) we have $A_1 : (1, 0, 0)$ and $B_3 : (1, 1, 1)$.
   - Let $A_1B_3 : [r_5, s_5, t_5]$.
   - A_1B_3 on $A_1 \implies 0 = [r_5, s_5, t_5] \cdot (1, 0, 0) = r_5$.
   - A_1B_3 on $B_3 \implies 0 = [0, s_5, t_5] \cdot (1, 1, 1) = s_5 + t_5 \implies t_5 = -s_5$.
   - Thus $A_1B_3 : [0, s_5, -s_5] = s_5[0, 1, -1]$.

   We summarize the above calculation by the notation:

$$A_1B_3 : [0, 1, -1] \quad \Longleftarrow \quad \left\langle \begin{array}{l} A_1 : (1, 0, 0) \\ B_3 : (1, 1, 1). \end{array} \right.$$

6. **The line A_2B_3.**
   Using items 2 and 4, and arguments similar to those used in item 5 give the result which is summarized by the notation:

$$A_2B_3 : [1, 0, -1] \Longleftarrow \left\langle \begin{array}{l} A_2 : (0, 1, 0) \\ B_3 : (1, 1, 1). \end{array} \right.$$

7. **The line A_3B_3.**
   Using items 3 and 4 and arguments similar to those used in item 5 we get the result summarized by the notation:

$$A_3B_3 : [1, -1, 0] \Longleftarrow \left\langle \begin{array}{l} A_3 : (0, 0, 1) \\ B_3 : (1, 1, 1). \end{array} \right.$$

8. **The point C_2 on A_1B_3.**
   - Let $C_2 : (x_8, y_8, z_8)$.
   - By (5), we have $A_1B_3 : [0, 1, -1]$.
   - C_2 on A_1B_3 implies $0{=}[0, 1, -1] \cdot (x_8, y_8, z_8) = y_8 - z_8$.
   - Thus $z_8 = y_8$ and $C_2 : (x_8, y_8, y_8)$.
   - $C_2 \neq A_1$ implies $y_8 \neq 0$ and hence y_8 has an inverse.
   - Let $a = x_8 y_8^{-1}$, so that $x_8 = a y_8$ and $C_2 : (a y_8, y_8, y_8) = (a, 1, 1) y_8$.
   - $C_2 \neq B_3$ implies $a \neq 1$.

   Thus we have

$$C_2 : (a, 1, 1), \text{ where } a \neq 1.$$

9. **The line A_3C_2.**
   - From item 3 we have $A_3 : (0, 0, 1)$
   - From item 8 we have $C_2 : (a, 1, 1)$.
   - Let $A_3C_2 : [r_9, s_9, t_9]$.
   - A_3C_2 on $A_3 \implies 0 = [r_9, s_9, t_9] \cdot (0, 0, 1) = t_9$.
   - Thus $A_3C_2 : [r_9, s_9, 0]$.
   - A_3C_2 on $C_2 \implies 0 = [r_9, s_9, 0] \cdot (a, 1, 1) = r_9 a + s_9$.
   - Thus $s_9 = -r_9 a$ and $[r_9, s_9, 0] = [r_9, -r_9 a, 0] = r_9[1, -a, 0]$.
   - We write $A_3C_2 : [1, -a, 0]$.

   We summarize the above with this notation:

$$A_3C_2 : [1, -a, 0] \quad \Longleftarrow \quad \left\langle \begin{array}{l} A_3 : (0, 0, 1) \\ C_2 : (a, 1, 1). \end{array} \right.$$

10. **The point C_1 on line A_2B_3.**
    In item 6 we saw that $A_2B_3 : [1, 0, -1]$.
    Following an argument analogous to that given in item 8, we find

$$C_1 : (1, b, 1), \text{ where } b \neq 1.$$

11. **The line** A_3C_1.
Following an argument similar to the one used in item 9, we find

$$A_3C_1 : [b, -1, 0] \Longleftarrow \left\langle \begin{array}{l} A_3 : (0, 0, 1) \\ C_1 : (1, b, 1). \end{array} \right.$$

12. **The line** A_1C_1.
We have

- $A_1 : (1, 0, 0)$ by (1).
- $C_1 : (1, b, 1)$ by (10).

Again we follow steps analogous to those in item 9, which we summarize by writing:

$$A_1C_1 : [0, 1, -b] \Longleftarrow \left\langle \begin{array}{l} A_1 : (1, 0, 0) \\ C_1 : (1, b, 1). \end{array} \right. \tag{4}$$

13. **The point** $B_1 = A_1C_1 \cap A_3C_2$.

- B_1 is on lines $A_1C_1 : [0, 1, -b]$ and $A_3C_2 : [1, -a, 0]$.
- Let $B_1 : (x_{13}, y_{13}, z_{13})$.
- A_1C_1 on B_1 implies: $0 = [0, 1, -b] \cdot (x_{13}, y_{13}, z_{13}) = y_{13} - bz_{13}$. Thus $B_2 : (x_{13}, bz_{13}, z_{13})$.
- A_3C_2 on $B_1 \implies 0 = [1, -a, 0] \cdot (x_{13}, bz_{13}, z_{13}) = x_{13} - a(bz_{13})$. Thus $B_2 : (x_{13}, y_{13}, z_{13}) = (a(bz_{13}), bz_{13}, z_{13}) = (ab, b, 1)z_{13}$.

We write

$$B_1 : (ab, b, 1).$$

In summary:

$$\left. \begin{array}{r} A_1C_1 : [0, 1, -b] \\ A_3C_2 : [1, -a, 0] \end{array} \right\rangle \quad \Longrightarrow \quad B_1 : (ab, b, 1).$$

14. **The line** A_2C_2.
Following the method used in item 12 we have $A_2C_2 : [1, 0, -a]$.

$$A_2C_2 : [1, 0, -a] \quad \Longleftarrow \quad \left\langle \begin{array}{l} A_2 : (0, 1, 0) \\ C_2 : (a, 1, 1). \end{array} \right.$$

15. **The point** $B_2 = A_2C_2 \cap A_3C_1$.

- Let $B_2 : (x_{15}, y_{15}, z_{15})$.
- A_2C_2 on $B_2 \implies 0 = [1, 0, -a] \cdot (x_{15}, y_{15}, z_{15}) = x_{15} - az_{15}$.
- A_3C_1 on $B_2 \implies 0 = [b, -1, 0] \cdot (az_{15}, y_{15}, z_{15}) = b(az_{15}) - y_{15}$.

- Thus $B_2 : (x_{15}, y_{15}, z_{15}) = (az_{15}, b(az_{15}), z_{15}) = (a, ba, 1)z_{15}$.

In summary

$$\left.\begin{array}{l} A_2C_2 : [1, 0, -a] \\ A_3C_1 : [-b, 1, 0] \end{array}\right\rangle \implies B_2 : (a, ba, 1).$$

16. **The line** A_1B_2.

- From item 1 we have $A_1 : (1, 0, 0)$
- From item 15 we have $B_2 : (a, ba, 1)$.
- Let $A_1B_2 : [r_{16}, s_{16}, t_{16}]$.
- A_1B_2 on $A_1 \implies 0 = [r_{16}, s_{16}, t_{16}] \cdot (1, 0, 0) = r_{16}$.
- A_1B_2 on $B_2 \implies 0 = [0, s_{16}, t_{16}] \cdot (a, ba, 1) = s_{16}(ba) + t_{16}$.
- $A_1B_2 : [r_{16}, s_{16}, t_{16}] = [0, s_{16}, -s_{16}ba] = s_{16}[0, 1, -ba]$.
- We write $A_1B_2 : [0, 1, -ba]$.

In summary,

$$A_1B_2 : [0, 1, -ba] \impliedby \left\langle \begin{array}{l} A_1 : (1, 0, 0) \\ B_2 : (a, ba, 1). \end{array}\right.$$

17. **Find the line** A_2B_1.

- We have $A_2 : (0, 1, 0)$ and $B_1 : (ab, b, 1)$.
- Let $A_2B_1 : [r_{17}, s_{17}, t_{17}]$.
- A_2 on $A_2B_1 \implies 0 = [r_{17}, s_{17}, t_{17}] \cdot (0, 1, 0) = s_{17}$.
- B_1 on $A_2B_1 \implies 0 = [r_{17}, 0, t_{17}] \cdot (ab, b, 1) = r_{17}(ab) + t_{17}$.
- Thus $[r_{17}, s_{17}, t_{17}] = [r_{17}, 0, -r_{17}(ab)] = r_{17}[1, 0, -ab]$.
- We write: $A_2B_1 : [1, 0, -ab]$.

In summary,

$$A_2B_1 : [1, 0, -ab] \impliedby \left\langle \begin{array}{l} A_2 : (0,\ 1,\ 0) \\ B_1 : (ab, b, 1). \end{array}\right.$$

18. **The point** $C_3 := A_1B_2 \cap A_2B_1$**.**

- From (16) we have $A_1B_2 : [1, 0, -ab]$ and from (17) $A_2B_1 : [0, 1, -ba]$.
- Let $C_3 = (x_{18}, y_{18}, z_{18})$.
- C_3 on A_1B_2 implies $0 = [1, 0, -ab] \cdot (x_{18}, y_{18}, z_{18}) = x_{18} - abz_{18}$.
- C_3 on A_2B_1 implies $0 = [0, 1, -ba] \cdot (x_{18}, y_{18}, z_{18}) = y_{18} - baz_{18}$.
- $(x_{18}, y_{18}, z_{18}) = (abz_{18}, baz_{18}, z_{18}) = (ab, ba, 1)z_{18}$.

Thus

$$C_3 : (ab, ba, 1).$$

These steps are summarized in the notation

$$\left.\begin{array}{l} L_1 = A_1B_2 : [0, 1, -ab] \\ L_2 = A_2B_1 : [1, 0, -ba] \end{array}\right\rangle \implies C_3 : (ab, ba, 1).$$

19. **Is C_3 on A_3B_3?**

    - By item 7, we have $A_3B_3 : [1, -1, 0]$
    - By item 18 we have $C_3 = (ab, ba, 1)$.

    Then

$$A_3B_3 \text{ is on } C_3 \iff [1, -1, 0] \cdot (ab, ba, 1) = 0 \iff ab = ba.$$

In the case that the division ring is a field, we have the proof of Pappus's theorem.

But even better, we see that if the geometric configuration of Pappus's configuration always closes with 9 points and 9 lines, the coordinatizing division ring must be commutative.

This completes the proof. ∎