OPERATOR ALGEBRAS FOR MULTIVARIABLE DYNAMICS

KENNETH R. DAVIDSON AND ELIAS G. KATSOULIS

Abstract. Let X be a locally compact Hausdorff space with n proper continuous self maps $\sigma_i : X \to X$ for $1 \le i \le n$. To this we associate various topological conjugacy algebras; and two emerge as the natural candidates for the universal algebra of the system, the tensor algebra $\mathcal{A}(X,\sigma)$ and the semicrossed product $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$. The C*-envelope of $\mathcal{A}(X,\sigma)$ is the Cuntz-Pimsner algebra $\mathrm{C}^*(X,\sigma)$ as defined by Katsura.

We introduce a new concept of conjugacy for multidimensional systems, which we coin piecewise conjugacy. We prove that the piecewise conjugacy class of the system can be recovered from either the algebraic structure of $\mathcal{A}(X,\sigma)$ or $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$. Various classification results follow as a consequence. For example, for $n = 2, 3$, the tensor algebras are (algebraically or even completely isometrically) isomorphic if and only if the systems are piecewise topologically conjugate.

We define a generalized notion of wandering sets and recurrence. Using this, it is shown that $\mathcal{A}(X,\sigma)$ or $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ is semisimple if and only if there are no generalized wandering sets. In the metrizable case, this is equivalent to each σ_i being surjective and v-recurrent points being dense for each $v \in \mathbb{F}_n^+$.

Table of Contents

2000 *Mathematics Subject Classification.* 47L55, 47L40, 46L05, 37B20, 37B99.

Key words and phrases: multivariable dynamical system, operator algebra, tensor algebra, semi-crossed product, Cuntz-Pimsner C*-algebra, semisimple, radical, piecewise conjugacy, wandering sets, recurrence.

First author partially supported by an NSERC grant.

Second author was partially supported by a grant from ECU.

1. Introduction

Let X be a locally compact Hausdorff space; and suppose we are given n proper continuous self maps $\sigma_i : X \to X$ for $1 \le i \le n$, i.e., a *multivariable dynamical system*. This paper is the first attempt to develop a theory of conjugacy algebras for such multivariable dynamical systems, with one of the goals being the association of the dynamics of the multivariable systems to fundamental areas of study within operator algebra theory. One of the main outcomes of this work is that the classification and representation theory of conjugacy algebras is intimately connected to piecewise conjugacy and generalized recurrence for multivariable systems. These concepts appear to be new in multivariable dynamical system theory and are first studied in this paper.

We discover a variety of new operator algebras. For the system of a single map σ, there is a single prototypical example, the semicrossed product of the system, first introduced by Arveson [**1**] but formally studied by Peters [**37**]. As we shall see, the case $n > 1$ offers a far greater diversity of examples and it seems that there are various non-isomorphic algebras that can serve as a prototype for the conjugacy algebra of the system. The algebras will contain an isometric copy of $\mathrm{C}_0(X)$. Plus they need to contain generators $\mathfrak{s}_i$ which encode the covariance relations of the maps σ_i. In addition, it is necessary to impose norm conditions to be able to talk about a *universal operator algebra* for the system. The choice of these conditions creates more than one natural universal operator algebra. Ideally, the properties of the algebra should reflect properties of the system. Based on that, we isolate two natural choices for the appropriate universal operator algebra for the system.

The two choice that we focus on are the case in which the generators are either isometric, producing the *semicrossed product*, or row isometric, producing the *tensor algebra*. We develop an appropriate dilation theory for each. This is both more straightforward and more satisfying in the case of the tensor algebra, where it turns out to be a C*-correspondence in the sense of Muhly and Solel [**31**]. This enables us to exploit their work, and work of Katsura [**23**] and Katsoulis–Kribs [**22**], in order to describe the C*-envelope of the tensor algebra.

In the semicrossed product situation, one needs to work harder to achieve what we call a *full dilation*. This turns out to be both orthogonally injective and projective, and by another result of Muhly and Solel [**30**], these representations turn out to be boundary representations in the sense of Arveson [**2**] except that they need not be irreducible. This allows us to show that generally these algebras are not C*-correspondences.

In [**18**], Hadwin and Hoover considered a rather general class of conjugacy algebras associated to a single dynamical system. In [**9**], we refined these axioms for the Banach algebra category. By viewing conjugacy algebras as equivalent if they arise from conjugate systems, we were able to show that the conjugacy class of the system forms a complete invariant for algebraic isomorphisms between equivalence classes of conjugacy algebras. In this paper, we are taking a more restrictive view in order to focus on the most natural candidates. Nevertheless, the ideas of [**9**] will play a crucial role in the analysis of our algebras.

It is also natural to ask that there be an expectation of these operator algebras onto the diagonal $C_0(X)$. In both cases which we consider, such an expectation is available. This leads, among other things, to the notion of a Fourier series for elements of these algebras. This is a useful tool for establishing automatic continuity of algebraic isomorphisms among this class of algebras.

In Section 7, following the techniques of Hadwin–Hoover [**18**] and our techniques in [**9**] for the one variable case, we describe the characters and the nest representations into the algebra $\mathfrak{T}_2$ of 2×2 upper triangular matrices. In Section 8 we introduce a new concept in dynamical system theory, *piecewise conjugacy*, and we study its elementary properties. The results in both these sections are mostly preparatory for the results of the subsequent section.

In Section 9, we prove the classification theorems. Working in the multivariable setting, one often comes up against issues in multivariable algebra or analysis. Here we find that we need to exploit properties of analytic varieties. Theorem 9.2 allows us to recover the dynamical system up to piecewise conjugacy. Essentially one can determine the system locally, that is, on some neighbourhood of each point of X; but the pairing between the two systems can change from one open set to the next. This works on *both* the tensor algebra and the semi-crossed product. In order to establish the converse in certain cases, we find that we need topological information about the unitary group which limits the dimension of systems where we are able to obtain a complete invariant. However for $n = 2$ or 3, or when X is 0 or 1 dimensional, we show that piecewise conjugacy is a complete invariant up to either algebraic or completely isometric isomorphism. We state a conjecture regarding the general situation.

In the case of a single map, the radical of the semicrossed product has been studied [**28, 37**] and finally was completely characterized by

Donsig, Katavolos and Manoussos [**10**] using a generalized notion of recurrence. In Section 10, we introduce a generalized notion of wandering set for a dynamical system which is appropriate for a non-commutative multivariable setting such as ours. This is related to a generalized notion of recurrence, and are equivalent in the metrizable setting.

These topological notions are exploited in the last section to characterize when the tensor algebra and the semicrossed product are semisimple in terms of the dynamics. The more general issue of describing the radical remains an interesting open problem.

Finally we mention that the ideas of this paper may be applied to dynamical systems with relations among the continuous maps. This issue will be explored in another paper.

2. Universal Operator Algebras

We now discuss the choice of an appropriate covariance algebra for the multivariable dynamical system (X,σ). An operator algebra encoding (X,σ) should contain $\mathrm{C}_0(X)$ as a C*-subalgebra, and there should be n elements s_i satisfying the *covariance relations*

$$fs_i = s_i(f\circ\sigma_i) \quad \text{for} \quad f\in \mathrm{C}_0(X) \text{ and } 1\le i\le n.$$

This relation shows that $s_{i_k}f_k s_{i_{k-1}}f_{k-1}\dots s_{i_1}f_1 = s_w g$ where we write $s_w = s_{i_k}s_{i_{k-1}}\dots s_{i_k}$ and g is a certain product of the f_j's composed with functions built from the σ_i's. Thus the set of polynomials in $s_1,\dots,s_n$ with coefficients in $\mathrm{C}_0(X)$ forms an algebra which we call the *covariance algebra* $\mathcal{A}_0(X,\sigma)$. The universal algebra should be the (norm-closed non-selfadjoint) operator algebra obtained by completing the covariance algebra in an appropriate operator algebra norm.

Observe that in the case of compact X, $\mathcal{A}_0(X,\sigma)$ is unital, and will contain the elements s_i as generators. When X is not compact, it is generated by $\mathrm{C}_0(X)$ and elements of the form s_if for $f\in\mathrm{C}_0(X)$.

By an operator algebra, we shall mean an algebra which is completely isometrically isomorphic to a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. By the Blecher–Ruan–Sinclair Theorem [**7**], there is an abstract characterization of such algebras. See [**6, 36**] for a thorough treatment of these issues. Our algebras are sufficiently concrete that we will not need to call upon these abstract results. Nevertheless, it seems more elegant to us to define universal operator algebras abstractly rather than in terms of specific representations.

An operator algebra claiming to be *the* operator algebra of the system must be universal in some way. This requires a choice of an appropriate norm condition on the generators. A few natural choices are:

(1) **Contractive:** $\|s_i\| \le 1$ for $1 \le i \le n$.
(2) **Isometric:** $s_i^* s_i = I$ for $1 \le i \le n$.
(3) **Row Contractive:** $\left\| \begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix} \right\| \le 1$.
(4) **Row Isometric:** $\begin{bmatrix} s_1 & s_2 & \dots & s_n \end{bmatrix}$ is an isometry; i.e. $s_i^* s_j = \delta_{ij}$ for $1 \le i, j \le n$.

One could add variants such as unitary, co-isometric, column contractive, etc.

In the one variable case, all of these choices are equivalent. Indeed, the Sz.Nagy isometric dilation of a contraction is compatible with extending the representation of $\mathrm{C}_0(X)$. This leads to the semi-crossed product introduced by Peters [**37**]. Various non-selfadjoint algebras associated to a dynamical system (with one map) have been studied [**1, 5, 26, 18, 40, 28, 10**].

But once one goes to several variables, these notions are distinct, even in the case of commutative systems. For example, with three or more commuting variables, examples of Varopoulos [**42**] and Parrott [**35**] show that three commuting contractions need not dilate to three commuting isometries. However a dilation theorem of Drury [**13**] does show that a strict row contraction of n commuting operators dilates to (a multiple of) Arveson's d-shift [**3**]. While this is not an isometry, it is the appropriate universal commuting row contraction.

For non-commuting variables, where there is no constraint such as commutativity, one could dilate the n contractions to isometries separately. We shall see that this can be done while extending the representation of $\mathrm{C}_0(X)$ to maintain the covariance relations. Also for the row contraction situation, there is the dilation theorem of Frahzo–Bunce–Popescu [**14, 8, 39**] which allows dilation of any row contraction to a row isometry. Again we shall show that this can be done while extending the representation of $\mathrm{C}_0(X)$ to preserve the covariance relations.

Definition 2.1. A locally compact Hausdorff space X together with n proper continuous maps σ_i of X into itself for $1 \le i \le n$ will be denoted by (X, σ). We shall refer to this as a *multivariable dynamical system.* It will be called *metrizable* if X is metrizable.

We now define the two universal operator algebras which we will associate to (X, σ). We justify the nomenclature below.

Definition 2.2. Given a multivariable dynamical system (X, σ), define the **tensor algebra** to be the universal operator algebra $\mathcal{A}(X, \sigma)$ generated by $\mathrm{C}_0(X)$ and generators $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ satisfying the covariance relations

$$f s_i = s_i (f \circ \sigma_i) \quad \text{for} \quad f \in \mathrm{C}_0(X) \text{ and } 1 \le i \le n$$

and satisfying the row contractive condition $\left\| \begin{bmatrix} \mathfrak{s}_1 & \mathfrak{s}_2 & \dots & \mathfrak{s}_n \end{bmatrix} \right\| \le 1$.

Similarly, we define the **semicrossed product** to be the universal operator algebra $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ generated by $\mathrm{C}_0(X)$ and generators $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ satisfying the covariance relations and satisfying the contractive condition $\|\mathfrak{s}_i\| \le 1$ for $1 \le i \le n$.

We will not belabour the set theoretic issues in defining a universal object like this, as these issues are familiar. Suffice to say that one can fix a single Hilbert space of sufficiently large dimension, say $\aleph_0|X|$, on which we consider representations of $\mathrm{C}_0(X)$ and the covariance relations. Then one puts the abstract operator algebra structure on $\mathcal{A}_0(X,\sigma)$ obtained by taking the supremum over all (row) contractive representations. Alternatively, one forms the concrete operator algebra by taking a direct sum over all such representations on this fixed space.

A case can be made for preferring the row contraction condition, based on the fact that this algebra is related to other algebras which have been extensively studied in recent years. If X is a countable discrete set, then the row contractive condition yields the graph algebra of the underlying directed graph that forgets which map σ_i is responsible for a given edge from x to $\sigma_i(x)$. In the general case, this turns out to be a C*-correspondence algebra, or tensor algebra, as defined by Muhly and Solel [**31**]. It is for this reason that we call this algebra the tensor algebra of the dynamical system. As such, it sits inside a related Cuntz–Pimsner C*-algebra [**38**], appropriately defined and studied by Katsura [**23**] building on an important body of work by Muhly and Solel beginning with [**31, 32**]. This Cuntz-Pimsner algebra turns out to be the C*-envelope [**2, 17**] of the tensor algebra [**31, 16, 20**]. The C*-envelope of the tensor algebra is therefore always nuclear.

We may consider the dynamical system (X,σ) as an action of the free semigroup $\mathbb{F}_n^+$. The free semigroup $\mathbb{F}_n^+$ consists of all words in the alphabet $\{1,2,\dots,n\}$ with the empty word $\varnothing$ as a unit. For each $w = i_k i_{k-1} \dots i_1$ in $\mathbb{F}_n^+$, let σ_w denote the map $\sigma_{i_k} \circ \sigma_{i_{k-1}} \circ \dots \circ \sigma_{i_1}$. This semigroup of endomorphisms of X induces a family of endomorphisms of $\mathrm{C}_0(X)$ by $\alpha_w(f) = f \circ \sigma_w$. The map taking $w \in \mathbb{F}_n^+$ to α_w is an antihomomorphism of $\mathbb{F}_n^+$ into $\mathrm{End}(\mathrm{C}_0(X))$; i.e. $\alpha_v \alpha_w = \alpha_{wv}$ for $v, w \in \mathbb{F}_n^+$.

This leads us to consider the contractive condition, which is the same as considering contractive covariant representations of the free semigroup. Hence we call the universal algebra the semi-crossed product $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ of the dynamical system. It also has good properties. However we do not find this algebra as tractable as the tensor algebra.

Indeed, several problems that are resolved in the tensor algebra case remain open for the semicrossed product. In particular, it often occurs (see Proposition 6.10) that the C*-envelope of the semicrossed product is not nuclear.

In both cases, the (row) contractive condition turns out to be equivalent to the (row) isometric condition. This is the result of dilation theorems to extend (row) contractive representations to (row) isometric ones. These are analogues of a variety of well-known dilation theorems. The tensor algebra case is easier than the semicrossed product, and in addition, there is a nice class of basic representations in this case that determine the universal norm. In the case of the crossed product, one needs to introduce the notion of a *full isometric dilation*; and these turn out to yield the (not necessarily irreducible) boundary representations of the C*-envelope.

3. Dilation for the Tensor Algebra

We first consider a useful family of representations for the tensor algebra analogous to those used by Peters **[37]** to define the semicrossed product of a one variable system.

By Fock space, we mean the Hilbert space $\ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$. This has the standard left regular representation of the free semigroup $\mathbb{F}_n^+$ defined by

$$L_v \xi_w = \xi_{vw} \quad \text{for} \quad v, w \in \mathbb{F}_n^+.$$

Consider the following *Fock space representations* of (X, σ). Fix x in X. Define a $*$-representation π_x of $\mathrm{C}_0(X)$ on $\mathcal{F}_x = \ell^2(\mathbb{F}_n^+)$ by $\pi_x(f) = \operatorname{diag}(f(\sigma_w(x)))$, i.e.

$$\pi_x(f)\xi_w = f(\sigma_w(x))\xi_w \quad \text{for} \quad f \in \mathrm{C}_0(X) \text{ and } w \in \mathbb{F}_n^+.$$

Send the generators s_i to L_i, and let $L_x = \begin{bmatrix} L_1 & \dots & L_n \end{bmatrix}$. Then (π_x, L_x) is easily seen to be a covariant representation.

Define the *full Fock representation* to be the (generally non-separable) representation $(\Pi, \mathbf{L})$ where $\Pi = \sum_{x\in X}^{\oplus} \pi_x$ and $\mathbf{L} = \sum_{x\in X}^{\oplus} L_x$ on $\mathcal{F}_X = \sum_{x\in X}^{\oplus} \mathcal{F}_x$. We will show that the norm closed algebra generated by $\Pi(\mathrm{C}_0(X))$ and $\Pi(\mathrm{C}_0(X))\mathbf{L}_i$ for $1 \le i \le n$ is completely isometric to the tensor algebra $\mathcal{A}(X, \sigma)$. When X is separable, a direct sum over a countable dense subset of X will yield a completely isometric copy on a separable space.

Now we turn to the dilation theorem, which is straight-forward given our current knowledge of dilation theory. When the dynamical system is surjective, this is closely related to **[31**, Theorem 3.3**]**.

Theorem 3.1. *Let (X,σ) denote a multivariable dynamical system. Let π be a $*$-representation of $\mathrm{C}_0(X)$ on a Hilbert space $\mathcal{H}$, and let $A = \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}$ be a row contraction satisfying the covariance relations*

$$\pi(f)A_i = A_i\pi(f\circ\sigma_i) \quad \textit{for} \quad 1 \le i \le n.$$

Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, a $*$-representation ρ of $\mathrm{C}_0(X)$ on $\mathcal{K}$ and a row isometry $\begin{bmatrix} S_1 & \dots & S_n \end{bmatrix}$ such that

(i) *$\rho(f)S_i = S_i\rho(f\circ\sigma_i)$ for $f \in \mathrm{C}_0(X)$ and $1 \le i \le n$.*
(ii) *$\mathcal{H}$ reduces ρ and $\rho(f)|_{\mathcal{H}} = \pi(f)$ for $f \in \mathrm{C}_0(X)$.*
(iii) *$\mathcal{H}^\perp$ is invariant for each S_i, and $P_{\mathcal{H}}S_i|_{\mathcal{H}} = A_i$ for $1 \le i \le n$.*

Proof. The dilation of A to a row isometry S is achieved by the Frahzo–Bunce–Popescu dilation [**14, 8, 39**]. Consider the Hilbert space $\mathcal{K} = \mathcal{H}\otimes\ell^2(\mathbb{F}_n^+)$ where we identify $\mathcal{H}$ with $\mathcal{H}\otimes\mathbb{C}\xi_\varnothing$. Following Bunce, consider A as an operator in $\mathcal{B}(\mathcal{H}^{(n)},\mathcal{H})$. Using the Schaeffer form of the isometric dilation, we can write $D = (I_{\mathcal{H}}\otimes I_n - A^*A)^{1/2}$ in $\mathcal{B}(\mathcal{H}^{(n)})$ and $I_{\mathcal{H}}\otimes L = \begin{bmatrix} I_{\mathcal{H}}\otimes L_1 & \dots & I_{\mathcal{H}}\otimes L_n \end{bmatrix}$. We make the usual observation that $(\mathbb{C}\xi_\varnothing)^\perp$ is identified with $\ell^2(\mathbb{F}_n^+)^{(n)}$ in such a way that $L_i|_{(\mathbb{C}\xi_\varnothing)^\perp} \simeq L_i^{(n)}$ for $1 \le i \le n$.

Then a (generally non-minimal) dilation is obtained as

$$S = \begin{bmatrix} A & 0 \\ JD & I_{\mathcal{H}}\otimes L^{(n)} \end{bmatrix}$$

where J maps $\mathcal{H}^{(n)}$ onto $\mathcal{H}\otimes\mathbb{C}^n \subset \mathcal{K}$ where the ith standard basis vector e_i in $\mathbb{C}^n$ is sent to ξ_i. Then

$$S_i = \begin{bmatrix} A_i & 0 \\ JD_i & I_{\mathcal{H}}\otimes L_i^{(n)} \end{bmatrix}$$

where $D_i = D|\mathcal{H}\otimes\mathbb{C}e_i$ is considered as an element of $\mathcal{B}(\mathcal{H},\mathcal{H}^{(n)})$.

To extend π, define a $*$-representation ρ on $\mathcal{K}$ by

$$\rho(f) = \mathrm{diag}(\pi(f\circ\sigma_w)).$$

That is,

$$\rho(f)(x\otimes\xi_w) = \pi(f\circ\sigma_w)x\otimes\xi_w \quad \text{for} \quad x\in\mathcal{H},\ w\in\mathbb{F}_n^+.$$

The restriction ρ_1 of ρ to $\mathcal{H}\otimes\mathbb{C}^n$ is just $\rho_1(f) = \mathrm{diag}(\pi(f\circ\sigma_i))$. The covariance relations for (π, A) may be expressed as

$$\pi(f)A = A\rho_1(f).$$

From this it follows that $\rho_1(f)$ commutes with A^*A and thus with D. In particular, $\rho_1(f)D_i = D_i\pi(f\circ\sigma_i)$. The choice of J then ensures that

$$\rho(f)S_i|_{\mathcal{H}\otimes\mathbb{C}\xi_\varnothing} = S_i|_{\mathcal{H}\otimes\mathbb{C}\xi_\varnothing}\pi(f\circ\sigma_i).$$

But the definition of ρ shows that

$$\rho(f)(I_{\mathcal{H}} \otimes L_i) = (I_{\mathcal{H}} \otimes L_i)\rho(f \circ \sigma_i)$$

Hence, as S_i agrees with $I_{\mathcal{H}} \otimes L_i$ on $\mathcal{H}^\perp = \mathcal{H} \otimes (\mathbb{C}\xi_\varnothing)^\perp$, we obtain

$$\rho(f)S_i|_{\mathcal{H}^\perp} = S_i\pi(f \circ \sigma_i)|_{\mathcal{H}^\perp} = S_i|_{\mathcal{H}^\perp}\pi(f \circ \sigma_i)|_{\mathcal{H}^\perp}.$$

Combining these two identities yields the desired covariance relation for (ρ, S).

The other properties of the dilation are standard. ∎

Remark 3.2. If one wishes to obtain the minimal dilation, one restricts to the smallest subspace containing $\mathcal{H}$ which reduces ρ and each S_i. The usual argument establishes uniqueness.

Corollary 3.3. *Every row contractive representation of the covariance algebra dilates to a row isometric representation.*

We now relate this to the special Fock space representations. It was an observation of Bunce [**8**] that the dilation S of A is pure if $\|A\| = r < 1$, where pure means that S is a multiple of the left regular representation L. In this case, the range $\mathcal{N}_0$ of the projection $P_0 = I - \sum_{i=1}^n S_iS_i^*$ is a cyclic subspace for S.

Observe that for any $f \in \mathrm{C}_0(X)$,

$$\begin{aligned}\rho(f)S_iS_i^* &= S_i\rho(f \circ \sigma_i)S_i^* = S_i\big(S_i\rho(\overline{f} \circ \sigma_i)\big)^* \\ &= S_i\big(\rho(f)S_i\big)^* = S_iS_i^*\rho(f).\end{aligned}$$

So P_0 commutes with ρ. Define a $*$-representation of $\mathrm{C}_0(X)$ by $\rho_0(f) = \rho(f)|_{\mathcal{N}_0}$.

Then we can recover ρ from ρ_0 and the covariance relations. Indeed, $\mathcal{K} = \sum_{w\in\mathbb{F}_n^+}^{\oplus} \mathcal{N}_w$ where $\mathcal{N}_w = S_w\mathcal{N}_0$. We obtain

$$\begin{aligned}\rho(f)P_{\mathcal{N}_w} &= \rho(f)S_wP_0S_w^* = S_w\rho(f \circ \sigma_w)P_0S_w^* \\ &= S_w\rho_0(f \circ \sigma_w)S_w^*.\end{aligned}$$

The spectral theorem shows that ρ_0 is, up to multiplicity, a direct integral of point evaluations. Thus it follows that the representation (ρ, S) is, in a natural sense, the direct integral of the special Fock space representations. Thus its norm is dominated by the norm of the full Fock representation.

As a consequence, we obtain:

Corollary 3.4. *The full Fock representation is a faithful completely isometric representation of the tensor algebra $\mathcal{A}(X, \sigma)$.*

Proof. By definition, if $T = \sum_{w\in\mathbb{F}_n^+} s_w f_w$ belongs to $\mathcal{A}_0(X,\sigma)$ (i.e. $f_w = 0$ except finitely often), its norm in $\mathcal{A}(X,\sigma)$ is determined as

$$\|T\|_\sigma := \sup \Big\| \sum_{w\in\mathbb{F}_n^+} A_w \pi(f_w) \Big\|$$

over the set of all row contractive representations (π, A). Clearly, we can instead sup over the set (π, rA) for $0 < r < 1$; so we may assume that $\|A\| = r < 1$. Then arguing as above, we see that (π, A) dilates to a row isometric representation (ρ, S) which is a direct integral of Fock representations. Consequently the norm

$$\Big\| \sum_{w\in\mathbb{F}_n^+} A_w \pi(f_w) \Big\| \le \Big\| \sum_{w\in\mathbb{F}_n^+} S_w \rho(f_w) \Big\| \le \Big\| \sum_{w\in\mathbb{F}_n^+} \mathbf{L}_w \Pi(f_w) \Big\|.$$

Thus the full Fock representation is completely isometric, and in particular is faithful. ∎

Remark 3.5. Indeed, the same argument shows that a faithful completely isometric representation is obtained whenever ρ_0 is a faithful representation of $\mathrm{C}_0(X)$. Conversely a representation ρ_0 on $\mathcal{H}$ induces a Fock representation ρ on $\mathcal{K} = \mathcal{H} \otimes \ell^2(\mathbb{F}_n^+)$ by $\rho(f) = \operatorname{diag}(\rho_0(f \circ \sigma_w))$. Then sending each s_i to $I_\mathcal{H} \otimes L_i$ yields a covariant representation which is faithful if ρ_0 is.

So now we can make some preliminary comments on the C*-envelope $\mathrm{C}^*(X,\sigma)$ of $\mathcal{A}(X,\sigma)$. We denote the generators which are the images of s_i by $\mathfrak{s}_i$, and we consider $\mathrm{C}_0(X)$ as a subalgebra. This is a quotient of the full Fock representation. Consequently, the generators determine an isometry $\mathfrak{s} = \begin{bmatrix} \mathfrak{s}_1 & \dots & \mathfrak{s}_n \end{bmatrix}$. Thus the isometries $\mathfrak{s}_i$ have pairwise orthogonal range. So they satisfy $\mathfrak{s}_j^* \mathfrak{s}_i = \delta_{ij}$. It is routine to verify now that the elements which are finite sums of the form $\sum_{v,w\in\mathbb{F}_n^+} \mathfrak{s}_v f_{v,w} \mathfrak{s}_w^*$ form a dense $*$-subalgebra of $\mathrm{C}^*(X,\sigma)$.

4. Fourier Series and Automatic Continuity

Since the tensor algebra has a universal property, it is evident that whenever (π, S) satisfies the covariance relations and the row contractivity $\big\| \begin{bmatrix} S_1 & \dots & S_n \end{bmatrix} \big\| \le 1$, then so does $(\pi, \lambda S)$ for $\lambda = (\lambda_i) \in \mathbb{T}^n$, where $\lambda S = \begin{bmatrix} \lambda_1 S_1 & \dots & \lambda_n S_n \end{bmatrix}$. Therefore the map α_λ which sends the generators $\mathfrak{s}_i$ of $\mathcal{A}(X,\sigma)$ to $\lambda_i \mathfrak{s}_i$ and fixes $\mathrm{C}_0(X)$ extends to a $*$-automorphism of $\mathrm{C}^*(X,\sigma)$ which fixes $\mathcal{A}(X,\sigma)$. In particular, if $\lambda_i = z \in \mathbb{T}$ for $1 \le i \le n$, we obtain the gauge automorphisms γ_z that are a key tool in many related studies.

One immediate application is standard:

Proposition 4.1. *The map* $E(a) = \displaystyle\int_{\mathbb{T}} \gamma_z(a)\,dz$ *is a completely contractive expectation of* $\mathcal{A}(X,\sigma)$ *onto* $\mathrm{C}_0(X)$*.*

Proof. It is routine to check that the map taking z to $\gamma_z(a)$ is norm continuous. Check this on $\mathcal{A}_0(X,\sigma)$ first and then approximate. So $E(a)$ makes sense as a Riemann integral. Now computing E on the monomials $\mathfrak{s}_w f$ shows that $E(f) = f$ and $E(\mathfrak{s}_w f) = 0$ for $w \neq \varnothing$. As this map is the average of completely isometric maps, it is completely contractive. ∎

Observe that in the Fock representation, one can see the expectation as the compression to the diagonal. The map E as defined above makes sense for any element of $\mathrm{C}^*(X,\sigma)$. However the range is then not $\mathrm{C}_0(X)$, but rather the span of all words of the form $\mathfrak{s}_v f \mathfrak{s}_w^*$ for $|v| = |w|$. We will make use of this extension below. It is also clear from this representation that one can read off the *Fourier coefficients* of elements of $\mathcal{A}(X,\sigma)$. This is computed within the universal C*-algebra as follows:

Definition 4.2. For each word $w \in \mathbb{F}_n^+$, define a map E_w from $\mathcal{A}(X,\sigma)$ onto $\mathrm{C}_0(X)$ by $E_w(a) = E(\mathfrak{s}_w^* a)$.

Observe that $\mathfrak{s}_w^*\mathfrak{s}_v = \mathfrak{s}_u$ when $v = wu$, it equals $\mathfrak{s}_u^*$ when $w = vu$, and otherwise it is 0. Thus the only time that $E(\mathfrak{s}_w^*\mathfrak{s}_v f) \neq 0$ is when $v = w$. Hence if a polynomial $a = \sum_{v\in\mathbb{F}_n^+} \mathfrak{s}_v f_v \in \mathcal{A}_0(X,\sigma)$, it is clear that $E_w(a) = f_w$.

These Fourier coefficients do not seem to be obtainable from an integral using invariants of $\mathcal{A}(X,\sigma)$ without passing to the C*-algebra. This means that they are less accessible than in the singly generated case. A partial recovery is the following. For $k \geq 0$, define

$$\Phi_k(a) = \int_{\mathbb{T}} \gamma_z(a)\overline{z}^k\,dz.$$

This is clearly a completely contractive map. Checking it on monomials shows that

$$\Phi_k(\mathfrak{s}_w f) = \begin{cases} \mathfrak{s}_w f & \text{if} \quad |w| = k \\ 0 & \text{if} \quad |w| \neq k \end{cases}.$$

Indeed, one can do somewhat better and obtain a sum over all words with the same abelianization. That is, if $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}_0^n$, define

$$\Psi_{\mathbf{k}}(a) = \int_{\mathbb{T}^n} \alpha_\lambda(a)\overline{\lambda}^{\mathbf{k}}\,d\lambda$$

where $\overline{\lambda}^{\mathbf{k}} = \prod_{i=1}^n \overline{\lambda}_i^{k_i}$. Again it is easy to check that this is a completely contractive map onto the span of words $\mathfrak{s}_w f$ such that $w(\lambda) = \lambda^{\mathbf{k}}$.

As for Fourier series, this series generally does not converge for arbitrary elements of $\mathcal{A}(X,\sigma)$. However one can define the Cesaro means and recover a from its Fourier series. Define the kth Cesaro mean by

$$\Sigma_k(a) = \sum_{i=0}^{k} \big(1 - \tfrac{i}{k}\big) \sum_{|w|=i} \mathfrak{s}_w E_w(a) = \sum_{i=0}^{k} \big(1 - \tfrac{i}{k}\big) \Phi_i(a).$$

As usual, this may be obtained as an integral against the Fejer kernel σ_k by

$$\Sigma_k(a) = \int_{\mathbb{T}} \gamma_z(a) \sigma_k(z)\, dz.$$

Since σ_k is positive with $\|\sigma_k\|_1 = 1$, this is again a completely contractive map. A routine modification of the usual Fejer Theorem of classical Fourier analysis shows that

$$a = \lim_{k\to\infty} \Sigma_k(a) \quad \text{for all} \quad a \in \mathcal{A}(X,\sigma).$$

So we may write $a \sim \sum_{w\in\mathbb{F}_n^+} \mathfrak{s}_w f_w$, where $f_w = E_w(a)$, to mean that this is the Fourier series of a, with summation interpreted via the Cesaro means.

A useful fact that derives from the Fourier series is the following:

Proposition 4.3. *Fix $k \geq 1$ and $a \in \mathcal{A}(X,\sigma)$. Suppose that $E_v(a) = 0$ for all $|v| < k$. Then a factors as $a = \sum_{|w|=k} \mathfrak{s}_w a_w$ for elements $a_w \in \mathcal{A}(X,\sigma)$; and $\|a\| = \big\| \sum_{|w|=k} a_w^* a_w \big\|^{1/2}$.*

Proof. First suppose that $a \in \mathcal{A}_0(X,\sigma)$. Observe that $a_w := \mathfrak{s}_w^* a$ belongs to $\mathcal{A}(X,\sigma)$. It is then clear that

$$\sum_{|w|=k} \mathfrak{s}_w a_w = \big(\sum_{|w|=k} \mathfrak{s}_w \mathfrak{s}_w^* \big) a = a.$$

Moreover a factors as

$$a = \begin{bmatrix} \mathfrak{s}_{w_1} & \dots & \mathfrak{s}_{w_{n^k}} \end{bmatrix} \begin{bmatrix} a_{w_1} \\ \vdots \\ a_{w_{n^k}} \end{bmatrix}$$

where $w_1, \dots, w_{n^k}$ is any enumeration of the words of length k. Since $\begin{bmatrix} \mathfrak{s}_{w_1} & \dots & \mathfrak{s}_{w_{n^k}} \end{bmatrix}$ is an isometry, if follows that

$$\|a\| = \big\| \sum_{|w|=k} a_w^* a_w \big\|^{1/2}.$$

For an arbitrary $a \in \mathcal{A}(X,\sigma)$ with $E_v(a) = 0$ for all $|v| < k$, note that its Cesaro means have the same property. Hence we can similarly define $a_w = \mathfrak{s}_w^* a$ and verify the result by taking a limit using these polynomials. ∎

For an arbitrary element of $\mathcal{A}(X,\sigma)$, subtract off the Fourier series up to level $k-1$ and apply the proposition. One gets:

Corollary 4.4. *Fix $k \geq 1$ and $a \in \mathcal{A}(X,\sigma)$. Then a can be written as $a = \sum_{|v|<k} \mathfrak{s}_v E_v(a) + \sum_{|w|=k} \mathfrak{s}_w a_w$ for certain elements $a_w \in \mathcal{A}(X,\sigma)$.*

An improtant feature of the Fourier series expansion is that if all the Fourier coefficients of $a \in \mathcal{A}(X,\sigma)$ are zero, then $a = 0$. This is a key property in establishing automatic continuity for isomorphisms between tensor algebras.

Recall that if $\varphi : \mathcal{A} \to \mathcal{B}$ is an epimorphism between Banach algebras, then the *separating space* of φ is the two-sided closed ideal of $\mathcal{B}$ defined as

$$\mathcal{S}(\varphi) := \big\{ b \in \mathcal{B} : \exists \{a_n\}_{n\geq 1} \subseteq \mathcal{A} \text{ so that } a_n \to 0 \text{ and } \varphi(a_n) \to b \big\}.$$

Clearly the graph of φ is closed if and only if $\mathcal{S}(\varphi) = \{0\}$. Thus by the closed graph theorem, φ is continuous if and only if $\mathcal{S}(\varphi) = \{0\}$.

The following is an adaption of [**41**, Lemma 2.1] and was used in [**9**].

Lemma 4.5 (Sinclair)**.** *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an epimorphism between Banach algebras and let $\{b_k\}_{k\in\mathbb{N}}$ be any sequence in $\mathcal{B}$. Then there exists $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$,*

$$\overline{b_1 b_2 \dots b_k \mathcal{S}(\varphi)} = \overline{b_1 b_2 \dots b_{k+1} \mathcal{S}(\varphi)}$$

and

$$\overline{\mathcal{S}(\varphi) b_k b_{k-1} \dots b_1} = \overline{\mathcal{S}(\varphi) b_{k+1} b_k \dots b_1}.$$

Corollary 4.6. *Let (X,σ) and (Y,τ) be multivariable dynamical systems. Then any isomorphism $\gamma : \mathcal{A}(X,\sigma) \to \mathcal{A}(Y,\tau)$ is automatically continuous.*

Proof. Fix one of the generating isometries of $\mathcal{A}(Y,\tau)$, say $\mathfrak{t}_1$. For any subset $\mathcal{S}$ of $\mathcal{A}(Y,\tau)$, the faithfulness of the Fourier series expansion implies that

$$\bigcap_{k\geq 0} \mathfrak{t}_1^k \mathcal{S} = \{0\}.$$

Thus if $\mathcal{S}(\gamma) \neq \{0\}$, then taking $b_i = \mathfrak{t}_1$ in Lemma 4.5, we obtain an integer k_0 so that

$$\mathfrak{t}_1^{k_0} \mathcal{S}(\gamma) = \bigcap_{k\geq 0} \mathfrak{t}_1^k \mathcal{S}(\gamma) = \{0\}.$$

Since left multiplication by $\mathfrak{t}_1$ is injective, $\mathcal{S}(\gamma) = \{0\}$. Therefore γ is continuous. ∎

This result allows us to consider only *continuous* representations in the study of arbitrary isomorphisms between tensor algebras of multi-systems.

5. C*-correspondences and the C*-envelope

In this section, we identify the tensor algebra with the tensor algebra of a C*-correspondence in the sense of Pimsner [**38**], Muhly–Solel [**31**] and Katsura [**23**]. This will enable us to consider $\mathcal{A}(X, \sigma)$ as a subalgebra of a natural C*-algebra which is the C*-envelope of $\mathcal{A}(X, \sigma)$.

Define $E = X \times n$ to be the union of n disjoint copies of the space X. We will view $\mathcal{E} = \mathrm{C}_0(E)$ as a C*-correspondence over $\mathrm{C}_0(X)$. To this end, we need to define left and right actions of $\mathrm{C}_0(X)$ and define a $\mathrm{C}_0(X)$-valued inner product. Let $\xi = \xi(x, j)$ and η denote elements of $\mathcal{E}$, and $f \in \mathrm{C}_0(X)$. The actions are given by

$$\begin{aligned}(\xi \cdot f)(x, j) &= \xi(x, j) f(x) \\ (f \cdot \xi)(x, j) &:= \varphi(f)\xi(x, j) = \xi(x, j)(f \circ \sigma_j)(x)\end{aligned}$$

and the inner product is

$$\langle \xi \mid \eta \rangle(x) = \sum_{j=1}^{n} \overline{\xi(x, j)} \eta(x, j).$$

That is, $\mathcal{E}$ is a (right) Hilbert C*-module over $\mathrm{C}_0(X)$ [**25**] with the additional structure as a left module over $\mathrm{C}_0(X)$.

As a Hilbert C*-module, $\mathcal{E}$ has the operator space structure of column n-space $\mathrm{Col}_n(\mathrm{C}_0(X))$. The adjointable left multipliers of $\mathcal{E}$ form a C*-algebra $\mathfrak{L}(\mathcal{E})$. It will be convenient to explicitly identify the left action as a $*$-homomorphism φ of $\mathrm{C}_0(X)$ into $\mathfrak{L}(\mathcal{E})$ by

$$\varphi(f) = \mathrm{diag}(f \circ \sigma_1, \dots, f \circ \sigma_n).$$

We will usually write $\varphi(f)\xi$ rather than $f\xi$ to emphasize the role of φ, as is common practice.

In general, φ is not faithful. Indeed, set $U_0 = X \setminus \bigcup_{i=1}^{n} \sigma_i(X)$ which is open because the σ_i are proper. Then $\ker \varphi$ consists of all functions with support contained in U_0. However the Hilbert C*-module is full, since $\langle \mathcal{E}, \mathcal{E} \rangle = \mathrm{C}_0(X)$. Also the left action is essential, i.e. $\varphi(\mathrm{C}_0(X))\mathcal{E} = \mathcal{E}$.

We briefly review Muhly and Solel's construction of the tensor algebra of $\mathcal{E}$ and two related C*-algebras. Set $\mathcal{E}^{\otimes 0} = \mathrm{C}_0(X)$ and

$$\mathcal{E}^{\otimes k} = \underbrace{\mathcal{E} \otimes_{\mathrm{C}_0(X)} \mathcal{E} \otimes_{\mathrm{C}_0(X)} \cdots \otimes_{\mathrm{C}_0(X)} \mathcal{E}}_{k \text{ copies}} \quad \text{for} \quad k \geq 1.$$

Notice that $\xi f \otimes \eta = \xi \otimes \varphi(f)\eta$.

Let ε_i denote the column vector with a 1 in the ith position. A typical element of $\mathcal{E}$ has the form $\sum_{i=1}^n \varepsilon_i f_i$ for $f_i \in \mathrm{C}_0(X)$. For each word $w = i_k \dots i_1 \in \mathbb{F}_n^+$, write

$$\varepsilon_w := \varepsilon_{i_k} \otimes \cdots \otimes \varepsilon_{i_1}.$$

A typical element of $\mathcal{E}^{\otimes k}$ has the form $\sum_{|w|=k} \varepsilon_w f_w$ for $f_w \in \mathrm{C}_0(X)$.

Naturally, $\mathcal{E}^{\otimes k}$ is a $\mathrm{C}_0(X)$-bimodule with the rules

$$(\xi_k \otimes \cdots \otimes \xi_1) \cdot f = \xi_k \otimes \cdots \otimes (\xi_1 f)$$

and

$$f \cdot (\xi_k \otimes \cdots \otimes \xi_1) = (\varphi(f)\xi_k) \otimes \cdots \otimes \xi_1.$$

Observe that

$$\begin{aligned} f \cdot \varepsilon_w &= (\varphi(f)\varepsilon_{i_k}) \otimes \varepsilon_{i_{k-1}} \otimes \cdots \otimes \varepsilon_{i_1} \\ &= \varepsilon_{i_k}(f \circ \sigma_{i_k}) \otimes \varepsilon_{i_{k-1}} \otimes \cdots \otimes \varepsilon_{i_1} \\ &= \varepsilon_{i_k} \otimes \varepsilon_{i_{k-1}}(f \circ \sigma_{i_k} \circ \sigma_{i_{k-1}}) \otimes \cdots \otimes \varepsilon_{i_1} \\ &= \varepsilon_{i_k} \otimes \varepsilon_{i_{k-1}} \otimes \cdots \otimes \varepsilon_{i_1}(f \circ \sigma_{i_k} \circ \sigma_{i_{k-1}} \circ \cdots \circ \sigma_{i_1}) \\ &= \varepsilon_w(f \circ \sigma_w). \end{aligned}$$

This identifies a $*$-homomorphism φ_k of $\mathrm{C}_0(X)$ into $\mathfrak{L}(\mathcal{E}^{\otimes k})$ by

$$\varphi_k(f) = \operatorname{diag}(f \circ \sigma_w)_{|w|=k},$$

namely

$$\varphi_k(f) \sum_{|w|=k} \varepsilon_w g_w = \sum_{|w|=k} \varepsilon_w (f \circ \sigma_w) g_w.$$

The inner product structure is defined recursively by the rule

$$\langle \xi \otimes \mu, \eta \otimes \nu \rangle = \langle \mu, \varphi(\langle \xi, \eta \rangle)\nu \rangle \quad \text{for all} \quad \xi, \eta \in \mathcal{E},\ \mu, \nu \in \mathcal{E}^{\otimes k}.$$

This seems complicated, but in our basis it is transparent:

$$\Big\langle \sum_{|w|=k} \varepsilon_w f_w, \sum_{|w|=k} \varepsilon_w g_w \Big\rangle = \sum_{|w|=k} \overline{f}_w g_w.$$

The Fock space of $\mathcal{E}$ is $\mathcal{F}(\mathcal{E}) = \sum_{n \geq 0}^{\oplus} \mathcal{E}^{\otimes n}$. This becomes a C*-correspondence as well, with the $\mathrm{C}_0(X)$-bimodule actions already defined on each summand, and the $\mathrm{C}_0(X)$-valued inner product obtained by declaring the summands to be orthogonal. In particular, this yields a $*$-isomorphism φ_∞ of $\mathrm{C}_0(X)$ into $\mathfrak{L}(\mathcal{F}(\mathcal{E}))$ by $\varphi_\infty(f)|_{\mathcal{E}^{\otimes k}} = \varphi_k(f)$.

There is a natural tensor action of $\xi \in \mathcal{E}$ taking $\mathcal{E}^{\otimes k}$ into $\mathcal{E}^{\otimes k+1}$ given by

$$T_\xi^{(k)}(\xi_1 \otimes \cdots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k.$$

Define T_ξ acting on $\mathcal{F}(\mathcal{E})$ by setting $T_\xi|_{\mathcal{E}^{\otimes k}} = T_\xi^{(k)}$. The tensor algebra $\mathcal{T}_+(\mathcal{E})$ of $\mathcal{E}$ is the norm-closed non-selfadjoint subalgebra generated by $\varphi_\infty(\mathrm{C}_0(X))$ and $\{T_\xi : \xi \in \mathcal{E}\}$. The C*-algebra generated by $\mathcal{T}_+(\mathcal{E})$ is called the Toeplitz C*-algebra $\mathcal{T}(\mathcal{E})$.

We will show that $\mathcal{T}_+(\mathcal{E})$ is completely isometrically isomorphic to $\mathcal{A}(X,\sigma)$. In addition, we shall show that the enveloping C*-algebra of $\mathcal{A}(X,\sigma)$ in the full Fock representation is $*$-isomorphic to $\mathcal{T}(\mathcal{E})$. Moreover, we will identify the quotient of $\mathcal{T}(\mathcal{E})$ which is the C*-envelope of $\mathcal{A}(X,\sigma)$. This will be an application of Katsura [**23**] and Katsoulis and Kribs [**22**]. To describe this quotient, we need some further description of the work of Katsura.

As usual, the set $\mathfrak{K}(\mathcal{E})$ of compact multipliers is the ideal of elements of $\mathfrak{L}(\mathcal{E})$ generated by the rank one elements $\theta_{\xi,\eta}$ for $\xi,\eta \in \mathcal{E}$ given by $\theta_{\xi,\eta}\zeta = \xi\langle \eta,\zeta\rangle$. In our case, $\varphi(\mathrm{C}_0(X))$ is contained in $\mathfrak{K}(\mathcal{E})$. To see this, given $f \in \mathrm{C}_0(X)$, factor $f \circ \sigma_i = g_i h_i$. Then for $\zeta = \sum_{j=1}^n \varepsilon_j k_j \in \mathcal{E}$,

$$\begin{aligned} \sum_{i=1}^n \theta_{\varepsilon_i g_i, \varepsilon_i \bar h_i}\zeta &= \sum_{i=1}^n \varepsilon_i g_i \langle \varepsilon_i \bar h_i, \zeta\rangle = \sum_{i=1}^n \varepsilon_i g_i h_i k_i \\ &= \sum_{i=1}^n \varepsilon_i (f\circ\sigma_i) k_i = \varphi(f)\zeta \end{aligned}$$

Katsura's ideal for a C*-correspondence $\mathcal{E}$ over a C*-algebra $\mathfrak{A}$ is defined as

$$\mathcal{J}_\mathcal{E} = \varphi^{-1}(\mathfrak{K}(\mathcal{E})) \cap \operatorname{ann}(\ker\varphi)$$

where $\operatorname{ann}(\mathcal{I}) = \{a \in \mathfrak{A} : ab = 0 \text{ for all } b \in \mathcal{I}\}$. In our setting, since $\ker\varphi$ consists of functions supported on U_0, $\mathcal{J}_\mathcal{E} = I(\overline{U_0})$, the ideal of functions vanishing on $\overline{U_0}$.

The space $\mathcal{F}(\mathcal{E})\mathcal{J}_\mathcal{E}$ becomes a Hilbert C*-module over $\mathcal{J}_\mathcal{E}$. Moreover $\mathfrak{K}(\mathcal{F}(\mathcal{E})\mathcal{J}_\mathcal{E})$ is spanned by terms of the form $\theta_{\xi f,\eta}$ where $\xi,\eta \in \mathcal{F}(\mathcal{E})$ and $f \in \mathcal{J}_\mathcal{E}$. He shows that this is an ideal in $\mathcal{T}(\mathcal{E})$. The quotient $\mathcal{O}(\mathcal{E}) = \mathcal{T}(\mathcal{E})/\mathfrak{K}(\mathcal{F}(\mathcal{E})\mathcal{J}_\mathcal{E})$ is the Cuntz–Pimsner algebra of $\mathcal{E}$. (Note that in Muhly–Solel [**31**], this is called the relative Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{J}_\mathcal{E},\mathcal{E})$. However the crucial role of this particular ideal $\mathcal{J}_\mathcal{E}$ is due to Katsura.)

A representation of a C*-correspondence $\mathcal{E}$ consists of a linear map t of $\mathcal{E}$ into $\mathcal{B}(\mathcal{H})$ and a $*$-representation π of $\mathrm{C}_0(X)$ on $\mathcal{H}$ such that

(1) $t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle)$ for all $\xi, \eta \in \mathcal{E}$

(2) $\pi(f)t(\xi) = t(\varphi(f)\xi)$ for all $f \in \mathrm{C}_0(X)$, $\xi \in \mathcal{E}$.

Such a representation is automatically a right module map as well. Moreover, when π is injective, t is automatically an isometry. Denote by $\mathrm{C}^*(\pi, t)$ the C*-algebra generated by $\pi(\mathrm{C}_0(X))$ and $t(\mathcal{E})$.

There is a universal C*-algebra $\mathcal{T}_{\mathcal{E}}$ generated by such representations, and Katsura [**23**, Prop.6.5] shows that the universal C*-algebra is isomorphic to $\mathcal{T}(\mathcal{E})$.

There is an induced $*$-representation of $\mathfrak{K}(\mathcal{E})$ given by

$$\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*.$$

Katsura shows that if $\pi(f)$ belongs to $\psi_t(\mathfrak{K}(\mathcal{E}))$, then $f \in \mathcal{J}_{\mathcal{E}}$ and $\pi(f) = \psi_t(\varphi(f))$. Then he introduces an additional property of a representation which he calls covariance. Since we already have a different and natural use for this term, we shall call such a representation *reduced* if it satisfies

(3) $\pi(f) = \psi_t(\varphi(f))$ for all $f \in \mathcal{J}_{\mathcal{E}}$.

There is again a universal C*-algebra $\mathcal{O}_{\mathcal{E}}$ for reduced representations of $\mathcal{E}$. This algebra is shown to be $*$-isomorphic to $\mathcal{O}(\mathcal{E})$ [**23**, Prop.6.5].

One says that $\mathrm{C}^*(\pi, t)$ admits a gauge action if there is a map β of the circle $\mathbb{T}$ into $\mathrm{Aut}(\mathrm{C}^*(\pi, t))$ such that $\beta_z(\pi(f)) = \pi(f)$ and $\beta_z(t(\xi)) = zt(\xi)$ for $f \in \mathrm{C}_0(X)$ and $\xi \in \mathcal{E}$. The universal algebras have this property automatically.

Every operator algebra is contained completely isometrically in a canonical minimal C*-algebra known as its C*-envelope. Muhly and Solel [**31**, Theorem 6.4] show that, when φ is injective, that the C*-envelope of $\mathcal{T}_+(\mathcal{E})$ is $\mathcal{O}(\mathcal{E})$. This was further analyzed in [**16**]. In the non-injective case, this is done by Katsoulis-Kribs [**20**].

Theorem 5.1. *Let (X, σ) be a multivariable dynamical system, and let $\mathcal{E}$ be the associated C*-correspondence. Then $\mathcal{A}(X, \sigma)$ is completely isometrically isomorphic to the tensor algebra $\mathcal{T}_+(\mathcal{E})$. Consequently, the C*-envelope of $\mathcal{A}(X, \sigma)$ is $*$-isomorphic to $\mathcal{O}(\mathcal{E})$.*

Proof. The point is to observe that the Fock representations give rise to representations of $\mathcal{E}$ which are sufficient to yield a faithful representation of $\mathcal{T}_{\mathcal{E}}$. Fix $x \in \mathcal{X}$. Define a representation of $\mathcal{E}$ on $\mathcal{F}_x = \ell^2(\mathbb{F}_n^+)$ by

$$\pi_x(f) = \mathrm{diag}(f(\sigma_w(x))) \quad \text{and} \quad t_x(\xi) = \sum_{i=1}^n L_i \pi_x(g_i)$$

for $f \in \mathrm{C}_0(X)$ and $\xi = \sum_{i=1}^n \varepsilon_i g_i \in \mathcal{E}$. It is routine to verify for $\eta = \sum_{i=1}^n \varepsilon_i h_i$ that

$$t_x(\xi)^* t_x(\eta) = \sum_{i=1}^n \pi_x(\overline{g}_i h_i) = \pi_x(\langle \xi, \eta \rangle)$$

and

$$\begin{aligned} \pi_x(f) t_x(\xi) &= \sum_{i=1}^n \pi_x(f) L_i \pi_x(g_i) \\ &= \sum_{i=1}^n L_i \pi_x(f \circ \sigma_i) \pi_x(g_i) = t_x(\varphi(f)\xi). \end{aligned}$$

Moreover it is also clear that the C*-algebra generated is exactly the Fock representation of $\mathcal{A}(X,\sigma)$ for the point x. Again we take the direct sum over all $x \in X$ (or a countable dense subset in the separable case) to obtain a full Fock representation (Π, T). The resulting C*-algebra admits a gauge action by conjugating on each Fock space $\mathcal{F}_x$ by the unitary operator $U_z = \mathrm{diag}(z^{|w|})$. Since the representation Π is obviously faithful, Katsura's Theorem 6.2 of **[23]** shows that we obtain a faithful representation of $\mathcal{T}_{\mathcal{E}}$ provided that

$$\mathcal{I}_{\Pi,T} := \{ f \in \mathrm{C}_0(X) : \Pi(f) \in \psi_T(\mathfrak{K}(\mathcal{E})) \} = 0.$$

To see this, observe that if $\xi = \sum_{i=1}^n \varepsilon_i g_i \in \mathcal{E}$ and $\eta = \sum_{i=1}^n \varepsilon_i h_i$, then

$$\psi_T(\theta_{\xi,\eta}) = \sum_{x\in X}^{\oplus} t_x(\xi) t_x(\eta)^* = \sum_{x \in X}^{\oplus} \sum_{i,j=1}^n L_i g_i \overline{h}_j L_j^*.$$

All of the vectors $\xi_{\varnothing,x} \in \mathcal{F}_x$ lie in the kernel of all of these maps. On the other hand, if $f \neq 0$, then $f(x) \neq 0$ for some x and $\pi_x(f)\xi_{\varnothing,x} = f(x)\xi_{\varnothing,x} \neq 0$. This establishes the claim.

Now the rest follows from the discussion preceding the theorem. ∎

Example 5.2. We conclude this section with an example to illuminate these ideas. Consider $X = [0,1]$ and let $\sigma_1(x) = x/3$ and $\sigma_2(x) = (2+x)/3$. This is an iterated function system. Therefore there is a unique non-empty compact subset Y such that $Y = \sigma_1(Y) \cup \sigma_2(Y)$. In this case, it is easily seen to be the Cantor set X_∞.

We form the algebra $\mathcal{A}(X,\sigma)$ and the C*-correspondence $\mathcal{E}$. Observe that the kernel $\ker\varphi = I\big([0,\frac13] \cup [\frac23, 1]\big)$. Thus $\mathcal{J}_{\mathcal{E}} = I([\frac13, \frac23])$.

Katsura shows that $\varphi_\infty(\mathrm{C}_0(X))$ has $\{0\}$ intersection with the ideal $\mathfrak{K}(\mathcal{F}(\mathcal{E})\mathcal{J}_{\mathcal{E}})$. Hence the representation of $\mathrm{C}_0(X)$ into $\mathcal{T}(\mathcal{E})/\mathfrak{K}(\mathcal{F}(\mathcal{E})\mathcal{J}_{\mathcal{E}})$

is injective. However $\varphi_\infty(\mathrm{C}_0(X))$ does intersect $\mathfrak{K}(\mathcal{F}(\mathcal{E}))$. To see this, note that $\varphi_\infty(f) = \operatorname{diag}(\varphi^{(k)}(f))$ belongs to $\mathfrak{K}(\mathcal{F}(\mathcal{E}))$ if and only if

$$\lim_{k\to\infty} \|\varphi^{(k)}(f)\| = 0.$$

This is because each $\varphi^{(k)}(f)$ belongs to $\mathfrak{K}(\mathcal{F}(\mathcal{E}))$ for every f. (This was noted above for $\mathcal{E}$, but works just as well for $\mathcal{E}^{\otimes k}$.) Let $X_k = \bigcup_{|w|=k} \sigma_w(X)$. This is a decreasing sequence of compact sets with $\bigcap_{k\geq 1} X_k = X_\infty$. Thus it is easy to see that $\varphi_\infty^{-1}(\mathfrak{K}(\mathcal{F}(\mathcal{E}))) = I(X_\infty)$. In particular, the quotient of $\mathcal{T}(\mathcal{E})$ by $\mathfrak{K}(\mathcal{F}(\mathcal{E}))$ is not injective on $\mathrm{C}_0(X)$.

6. Dilation and the Semi-crossed Product

In this section, we consider the contractive case. Again a routine modification of the classical theory yields a dilation to the isometric case. This will allow us to determine certain faithful representations of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$.

Proposition 6.1. *Let (X,σ) denote a multivariable dynamical system. Let π be a $*$-representation of $\mathrm{C}_0(X)$ on a Hilbert space $\mathcal{H}$, and let $A_1,\dots,A_n$ be contractions satisfying the covariance relations*

$$\pi(f)A_i = A_i\pi(f\circ\sigma_i) \quad \textit{for} \quad 1 \leq i \leq n.$$

Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, a $*$-representation ρ of $\mathrm{C}_0(X)$ on $\mathcal{K}$ and isometries $S_1,\dots,S_n$ such that

(i) *$\rho(f)S_i = S_i\rho(f\circ\sigma_i)$ for $f\in \mathrm{C}_0(X)$ and $1\leq i\leq n$.*
(ii) *$\mathcal{H}$ reduces ρ and $\rho(f)|_{\mathcal{H}} = \pi(f)$ for $f \in \mathrm{C}_0(X)$.*
(iii) *$\mathcal{H}^\perp$ is invariant for each S_i, and $P_{\mathcal{H}}S_i|_{\mathcal{H}} = A_i$ for $1\leq i\leq n$.*

Proof. This time, we can dilate each isometry separately in the classical way so long as they use pairwise orthogonal subspaces for these extensions. Form $\mathcal{K} = \mathcal{H}\otimes \ell^2(\mathbb{F}_n^+)$, and again identify $\mathcal{H}$ with $\mathcal{H}\otimes\mathbb{C}\xi_\varnothing$. Let $D_i = (I - A_i^*A_i)^{1/2}$. As before, we make the identification of $(\mathbb{C}\xi_\varnothing)^\perp$ with $\ell^2(\mathbb{F}_n^+)^{(n)}$ in such a way that $L_i|_{(\mathbb{C}\xi_\varnothing)^\perp} \simeq L_i^{(n)}$ for $1\leq i\leq n$. Let $J_i = I_{\mathcal{H}}\otimes L_i|_{\mathbb{C}\xi_\varnothing}$ be the isometry of $\mathcal{H}\otimes\mathbb{C}\xi_\varnothing$ onto $\mathcal{H}\otimes\mathbb{C}\xi_i$ for $1\leq i\leq n$.

Define a $*$-representation ρ of $\mathrm{C}_0(X)$ as before by

$$\rho(f) = \operatorname{diag}(\pi(f\circ\sigma_w));$$

and define isometric dilations of the A_i by

$$V_i = \begin{bmatrix} A_i & 0 \\ J_iD_i & I_{\mathcal{H}}\otimes L_i^{(n)} \end{bmatrix}$$

Again we have identified $(\mathbb{C}\xi_\varnothing)^\perp$ with $\ell^2(\mathbb{F}_n^+)^{(n)}$.

To verify the covariance relations, compute: for $x \in \mathcal{H}$, $f \in \mathrm{C}_0(X)$ and $w \in \mathbb{F}_n^+ \setminus \{\varnothing\}$

$$\begin{aligned}\rho(f)V_i(x \otimes \xi_w) &= \rho(f)(x \otimes \xi_{iw}) \\ &= \pi(f \circ \sigma_i \circ \sigma_w)x \otimes \xi_{iw} \\ &= V_i(\pi(f \circ \sigma_i \circ \sigma_w)x \otimes \xi_w) \\ &= V_i\rho(f \circ \sigma_i)(x \otimes \xi_w).\end{aligned}$$

While if $w = \varnothing$,

$$\begin{aligned}\rho(f)V_i(x \otimes \xi_\varnothing) &= \rho(f)(A_i x \otimes \xi_\varnothing + D_i x \otimes \xi_i) \\ &= (\pi(f)A_i x) \otimes \xi_\varnothing + (\pi(f \circ \sigma_i)D_i x) \otimes \xi_i\end{aligned}$$

and

$$\begin{aligned}V_i\rho(f \circ \sigma_i)(x \otimes \xi_\varnothing) &= (A_i\pi(f \circ \sigma_i)x) \otimes \xi_\varnothing + (D_i\pi(f \circ \sigma_i)x) \otimes \xi_i \\ &= (\pi(f)A_i x) \otimes \xi_\varnothing + (D_i\pi(f \circ \sigma_i)x) \otimes \xi_i\end{aligned}$$

Thus we will have the desired relation provided that D_i commutes with $\pi(f \circ \sigma_i)$. This is true and follows from

$$A_i^* A_i \pi(f \circ \sigma_i) = A_i^* \pi(f) A_i = \pi(f \circ \sigma_i) A_i^* A_i. \qquad \blacksquare$$

Corollary 6.2. *Let (X, σ) denote a multivariable dynamical system. Every contractive covariant representation of (X, σ) dilates to an isometric representation.*

Let (π, S) be an isometric representation of (X, σ). The covariance relations extend to the abelian von Neumann algebra $\pi(\mathrm{C}_0(X))''$. This algebra has a spectral measure E_π defined on all Borel subsets of X. Indeed, there is a $*$-representation $\overline{\pi}$ of the C*-algebra $\mathrm{Bor}(X)$ of all bounded Borel functions on X extending π. For any Borel set $A \subset X$, the covariance relations say that

$$E_\pi(A)S_i = S_i E_\pi(\sigma_i^{-1}(A)).$$

or equivalently

$$E_\pi\big(\sigma_i^{-1}(A)\big) = S_i^* E_\pi(A) S_i.$$

However,

$$S_i E_\pi\big(\sigma_i^{-1}(A)\big) S_i^* = E_\pi(A) S_i S_i^* \le E_\pi(A).$$

This is not completely satisfactory.

Definition 6.3. An isometric covariant representation (π, S) is a *full isometric representation* provided that $S_i S_i^* = E_\pi(\sigma_i(X))$.

The calculation above shows that a full isometric representation has the following important property.

Lemma 6.4. *If (π, S) is a full isometric covariant representation, then*

$$S_i E_\pi\big(\sigma_i^{-1}(A)\big) S_i^* = E_\pi(A)$$

for all Borel subsets $A \subset \sigma_i(X)$.

The following result is a dilation in the spirit of unitary dilations, rather than isometric dilations, in that the new space contains $\mathcal{H}$ as a semi-invariant subspace (rather than a coinvariant subspace).

This result requires the existence of a Borel cross section for the inverse of a continuous map, and that appears to require a separability condition. We were not able to find any counterexamples for huge spaces, but the case of most interest is, in any case, the metrizable one. Recall that a locally compact Hausdorff space is metrizable if and only if it is second countable. In this case, it is also separable.

Theorem 6.5. *Let (X, σ) be a metrizable multivariable dynamical system. Every isometric covariant representation (π, S) of (X, σ) has a dilation to a full isometric representation in the sense that there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, a $*$-representation ρ of $\mathrm{C}_0(X)$ and isometries $T_1, \dots. T_n$ such that*

1. *(ρ, T) is a full isometric representation of (X, σ).*
2. *$\mathcal{H}$ reduces ρ and $\rho|_{\mathcal{H}} = \pi$, and*
3. *$\mathcal{H}$ is semi-invariant for each T_i and $P_{\mathcal{H}} T_i|_{\mathcal{H}} = S_i$.*

Proof. Choose a Borel selector ω_i for each σ_i^{-1}; that is, a Borel function ω_i taking $\sigma_i(X)$ to X so that $\sigma_i \circ \omega_i = \mathrm{id}_{\sigma_i(X)}$. The existence of such a function is elementary [**34**, Theorem 4.2].

Observe first that $S_i S_i^*$ commutes with π. For any $f \in \mathrm{C}_0(X)$,

$$\begin{aligned} \pi(f) S_i S_i^* &= S_i \pi(f \circ \sigma_i) S_i^* = S_i \big(S_i \pi(\overline{f} \circ \sigma_i)\big)^* \\ &= S_i \big(\pi(f) S_i\big)^* = S_i S_i^* \pi(f). \end{aligned}$$

The construction is recursive. Let $\mathcal{H}_i = E_\pi(\sigma_i(X))(I - S_i S_i^*)\mathcal{H}$, and let J_i be the natural injection of $\mathcal{H}_i$ into $\mathcal{H}$. Since $\mathcal{H}_i$ reduces $\overline{\pi}$, we may define a $*$-representation $\overline{\pi}_i$ as its restriction. Form a Hilbert space $\mathcal{L}_1 = \mathcal{H} \oplus \sum^{\oplus}_{1 \le i \le n} \mathcal{H}_i$. Define a $*$-representation π_1 on $\mathcal{L}_1$ by

$$\pi_1(f) = \pi(f) \oplus \sum^{\oplus}_{1 \le i \le n} \overline{\pi}_i(f \circ \omega_i)$$

Also define an extension $A_i^{(1)}$ of S_i by $A_i^{(1)}|_{\mathcal{H}} = S_i$, $A_i^{(1)}|_{\mathcal{H}_j} = 0$ for $j \neq i$ and $A_i^{(1)}|_{\mathcal{H}_i} = J_i$. Then since $f \circ \sigma_i \circ \omega_i|_{\sigma_i(X)} = f|_{\sigma_i(X)}$,

$$\begin{aligned} \pi_1(f)A_i^{(1)} &= \pi_1(f)(S_i + J_i) = \pi(f)S_i + \pi(f)J_iP_{\mathcal{H}_i} \\ &= S_i\pi(f \circ \sigma_i) + J_iP_{\mathcal{H}_i}E_\pi(\sigma_i(X))\pi(f) \\ &= S_i\pi(f \circ \sigma_i) + J_i\overline{\pi}_i(f \circ \sigma_i \circ \omega_i) \\ &= (S_i + J_i)\pi_1(f \circ \sigma_i) = A_i^{(1)}\pi_1(f \circ \sigma_i). \end{aligned}$$

So this is a contractive covariant representation of (X, σ). Hence it has an isometric dilation $(\rho_1, T^{(1)})$ on a Hilbert space $\mathcal{K}_1$. Moreover, by construction, the range of $T_i^{(1)}$ contains $E_\pi(\sigma_i(X))\mathcal{H}$.

Repeat this procedure with the representation $(\rho_1, T^{(1)})$ to obtain an isometric representation $(\rho_2, T^{(2)})$ such that the range of $T_i^{(2)}$ contains $E_{\rho_1}(\sigma_i(X))\mathcal{K}_1$. By induction, one obtains a sequence of dilations $(\rho_k, T^{(k)})$ on an increasing sequence of Hilbert spaces $\mathcal{K}_k$ so that, at every stage, the range of $T_i^{(k+1)}$ contains $E_{\rho_k}(\sigma_i(X))\mathcal{K}_k$.

Let $\mathcal{K}$ be the Hilbert space completion of the union of the $\mathcal{K}_i$. Define a $*$-representation ρ of $\mathrm{C}_0(X)$ on $\mathcal{K}$ by $\rho|_{\mathcal{K}_k} = \rho_k$, and set $T_i|_{\mathcal{K}_k} = T_i^{(k)}$. This is a full isometric dilation by construction. ∎

Douglas and Paulsen [**11**] promoted the view that the Hilbert space $\mathcal{H}$ on which a representation (π, S) is defined should be considered as a Hilbert module over $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$. Muhly and Solel [**29**] have adopted this view. We are concerned with two notions where we use their more suggestive nomenclature. A Hilbert module $\mathcal{K}$ over an operator algebra $\mathcal{A}$ is *orthogonally injective* if every contractive short exact sequence

$$0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{Q} \to 0$$

has a contractive splitting. Likewise a Hilbert module $\mathcal{Q}$ is *orthogonally projective* if every such contractive short exact sequence has a contractive splitting. These notions explain the significance of full dilations.

Proposition 6.6. *Let (π, A) be a contractive representation of the semicrossed product $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ on a Hilbert space $\mathcal{H}$, considered as a Hilbert $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ module.*

1. *$\mathcal{H}$ is orthogonally injective if and only if (π, A) is a full isometric representation.*
2. *$\mathcal{H}$ is orthogonally projective if and only if (π, A) is an isometric representation.*

Proof. Suppose that $\mathcal{H}$ is an orthogonally injective Hilbert module. Let (ρ, S) be a full dilation of (π, A) on a Hilbert space $\mathcal{M}$. The complement $\mathcal{Q} = \mathcal{H}^\perp$ together with the restriction $\tilde\rho$ of ρ and the compressions $T_i = P_\mathcal{Q} S_i|_\mathcal{Q}$ is the quotient Hilbert module. So

$$0 \to \mathcal{H} \to \mathcal{M} \to \mathcal{Q} \to 0$$

is a contractive short exact sequence of Hilbert modules. This splits, which implies that $\mathcal{Q}$ is invariant for each S_i. Fullness means that $S_iS_i^* = E_\rho(\sigma_i(X))$, which means that

$$A_iA_i^* \oplus T_iT_i^* = E_\pi(\sigma_i(X)) \oplus E_{\tilde\rho}(\sigma_i(X)).$$

From this it follows that each A_i is an isometry satisfying the fullness condition.

Conversely, suppose that (π, A) is a full isometric representation on $\mathcal{H}$. Let $0 \to \mathcal{H} \to \mathcal{M} \to \mathcal{Q} \to 0$ be a contractive short exact sequence of Hilbert modules. Let (ρ, S) be the representation associated to $\mathcal{M}$. Write $S_i = \begin{bmatrix} A_i & B_i \\ 0 & D_i \end{bmatrix}$ with respect to $\mathcal{M} = \mathcal{H} \oplus \mathcal{Q}$ (as Hilbert spaces). Then

$$\begin{aligned} E_\pi(\sigma_i(X)) \oplus E_{\tilde\rho}(\sigma_i(X)) &= E_\rho(\sigma_i(X)) \\ &\geq S_iS_i^* = (A_iA_i^* + B_iB_i^*) \oplus D_iD_i^*. \end{aligned}$$

The fullness of (π, S) asserts that $E_\pi(\sigma_i(X)) = A_iA_i^*$. Hence $B_iB_i^* = 0$. Thus $\mathcal{Q}$ is invariant for S and so the natural injection of $\mathcal{Q}$ into $\mathcal{M}$ is a contractive module map. That is, $\mathcal{H}$ is orthogonally injective.

The second part is similar. If (π, A) is not isometric, then an isometric dilation does not split. Conversely, if (π, A) is isometric and $0 \to \mathcal{K} \to \mathcal{M} \to \mathcal{H} \to 0$ is a contractive short exact sequence, then again we write $S_i = \begin{bmatrix} A_i & 0 \\ C_i & D_i \end{bmatrix}$ with respect to $\mathcal{M} = \mathcal{H} \oplus \mathcal{K}$ (as Hilbert spaces). The fact that A_i are isometries and S_i is a contraction ensures that $C_i = 0$ and thus the sequence splits. ∎

Muhly and Solel [**30**] relate this notion to Arveson's approach to dilation theory [**2**]. Any contractive representation of an algebra $\mathcal{A}$ dilates to a $*$-representation of the C*-envelope $\mathrm{C}^*(\mathcal{A})$ by Arveson's and Stinespring's Theorems. It is easy to see that a representation which is orthogonally injective is, in fact, the restriction of a $*$-representation. The converse is not true in general.

For an operator algebra $\mathcal{A}$ contained in a C*-algebra $\mathfrak{A}$ which it generates, Arveson [**2**] called an *irreducible* representation π of $\mathfrak{A}$ a *boundary representation* if the restriction to $\pi|_\mathcal{A}$ has a unique completely positive extension to $\mathfrak{A}$, namely the representation itself. In some literature, the condition of irreducibility has been dropped. Muhly and

Solel proved that a representation of $\mathcal{A}$ is a (possibly reducible) boundary representation if and only if it is both orthogonally injective and orthogonally projective. Dritschel and McCullough [**12**] showed that the direct sum of all (possibly reducible) boundary representations yields a faithful representation of the C*-envelope, $\mathrm{C}^*(\mathcal{A})$.

Very recently, Arveson [**4**] has shown that these reducible boundary representations are, in fact, direct integrals of representations which are irreducible boundary representations almost everywhere. This completes the original program proposed in [**2**]. He also observes that Dritschel and McCullough use an approach of Agler exploiting the fact that π is both orthogonally injective and orthogonally projective exactly when it is a *maximal dilation*, meaning that whenever σ is a dilation of π, then σ decomposes as a direct sum $\pi \oplus \rho$. As our notion of full dilation is evidently this same maximality property, this explains why they turn out to be the (possibly reducible) boundary representations.

Thus the set of irreducible full isometric representations are sufficient to determine the C*-envelope of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$. We summarize this discussion as a corollary.

Corollary 6.7. *The C*-envelope of* $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ *can be obtained as* $\mathrm{C}^*(\rho(\mathcal{A}_0(X,\sigma)))$ *where* ρ *is the direct sum of all full isometric representations of* $\mathcal{A}_0(X,\sigma)$ *on a fixed Hilbert space of dimension* $\aleph_0|X|$. *If* X *is separable and metrizable, then a separable Hilbert space will suffice.*

Example 6.8. It is not clear to us how to write down a complete family of full isometric representations for the semicrossed product. But the theorem above suggests that one consider the *full atomic representations* as a natural candidate. For each $x \in X$, let $\mathcal{H}_x$ denote a non-zero Hilbert space such that

$$\sum_{y \in \sigma_i^{-1}(x)} \dim \mathcal{H}_y = \dim \mathcal{H}_x \quad \text{for all} \quad x \in X \text{ and } 1 \le i \le n.$$

Taking the dimension to be the

$$\aleph_0 \max\{|\sigma_i^{-1}(x)| : x \in X,\ 1 \le i \le n\}$$

for every $\mathcal{H}_x$ will suffice. Then for each $(x,i) \in X \times n$, select isometries $S_{i,y} \in \mathcal{B}(\mathcal{H}_y, \mathcal{H}_x)$ for each $y \in \sigma_i^{-1}(x)$ so that they have pairwise orthogonal ranges and

$$\textsc{sot}\!-\!\sum_{y \in \sigma_i^{-1}(x)} S_{i,y} S_{i,y}^* = I_{\mathcal{H}_x}.$$

Then define $T_i = \text{SOT–}\sum_{y\in X} S_{i,y}$. Define $\pi(f)|_{\mathcal{H}_y} = f(y) I_{\mathcal{H}_y}$. Then we have a full isometric representation of (X,σ).

Interesting separable representations can be found by taking a dense countable subset Y such that $\sigma_i(Y) \subset Y$ and for every $y \in Y \cap \sigma_i(X)$, $Y \cap \sigma_i^{-1}(x)$ is non-empty. Then build isometries as above.

Unfortunately, the general structure of such representations appears to be very complicated. This makes it difficult to describe the algebraic structure in the C*-envelope.

There is an expectation E from the semicrossed product onto $\mathrm{C}_0(X)$. This is established exactly as in Proposition 4.1. The isometries in $\mathrm{C}_0(X)\times_\sigma \mathbb{F}_n^+$ do not have nice relations, so one cannot define the Fourier coefficients as easily as in the tensor algebra case. One can, though, define the projections Φ_k onto the span of all words of length k in $T_1,\dots,T_n$. This is a finite dimensional subspace spanned by these words, which form a basis. So in principle, there is a Fourier series that can be determined in this way. Certainly the Cesaro means exist as nice integrals just as before. These are completely contractive maps into $\mathcal{A}_0(X,\sigma)$ which converge in norm for every element of $\mathrm{C}_0(X)\times_\sigma\mathbb{F}_n^+$. There is no analogue of Proposition 4.3. In particular, exactly the same proof as for Corollary 4.6 yields automatic continuity of isomorphisms:

Corollary 6.9. *Let (X,σ) and (Y,τ) be multivariable dynamical systems. Then any isomorphism of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ onto $\mathrm{C}_0(Y) \times_\tau \mathbb{F}_n^+$ is automatically continuous.*

Finally, we show that in general the algebra $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ does not arise as the tensor algebra of some C*-correspondence. This implies in particular that the computations of Section 5 have no analogues in the context of semicrossed products, when $n > 1$.

Proposition 6.10. *Let (X,σ), $\sigma = (\sigma_1,\sigma_2,\dots,\sigma_n)$, $n > 1$, be a multivariable dynamical system. Assume that the maps σ_i have a common fixed point. Let $\mathcal{E}$ be an arbitrary* C**-correspondence, over a* C**-algebra $\mathfrak{A}$, and let $\mathcal{T}_+(\mathcal{E})$ be the associated tensor algebra. Then, the semicrossed product $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ and the tensor algebra $\mathcal{T}_+(\mathcal{E})$ are not completely isometrically isomorphic.*

Proof. Let x_0 be a common fixed of the σ_i for $1 \le i \le n$. Consider the representation π of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ which sends $f \in \mathrm{C}_0(X)$ to $f(x_0)$ and sends $\mathfrak{s}_i$ to the generators U_i of $\mathrm{C}^*_r(\mathbb{F}_n)$, the reduced C*-algebra of the free group $\mathbb{F}_n$. This representation is full since each U_i is unitary. Therefore by Proposition 6.6, π is orthogonally injective and orthogonally projective. So by **[30]**, π has a unique completely positive extension to the $\mathrm{C}^*(\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+)$ and it is a $*$-representation. Thus $\mathrm{C}^*_r(\mathbb{F}_n)$

is a quotient of $\mathrm{C}^*(\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+)$. In particular, the C*-envelope is not nuclear.

On the other hand, suppose that $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ and $\mathcal{T}_+(\mathcal{E})$ were completely isometrically isomorphic. The diagonal of $\mathcal{T}_+(\mathcal{E})$, and hence the diagonal of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$, contains an isomorphic copy of $\mathfrak{A}$. Therefore $\mathfrak{A}$ is commutative. Consequently the associated Cuntz-Pimsner algebra $\mathcal{O}_\mathcal{E}$ is nuclear [**23**, Corollary 7.5]. By a result of Katsoulis and Kribs [**21**], $\mathcal{O}(\mathcal{E})$ is the C*-envelope of $\mathcal{T}_+(\mathcal{E})$.

Thus as the C*-envelope of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ is not nuclear, it is not a tensor algebra. ∎

For specific correspondences $\mathcal{E}$, one can draw stronger conclusions than that of the above Proposition. For instance we will see in Corollary 7.5 that under weaker hypotheses than Proposition 6.10, the semicrossed product $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ and the tensor algebra $\mathcal{A}(X, \sigma)$ are not isomorphic *as algebras.*

7. Characters and Nest Representations.

In this section, we extend methods from [**9**] to the multivariable setting. This will be applied in the Section 9 to recover much of the dynamical system from the tensor or crossed product algebra. Following Hadwin-Hoover [**18**], we first look at characters. At fixed points, there will be some analytic structure which will be important. Then we study nest representations into the 2×2 upper triangular matrices $\mathfrak{T}_2$.

In this section, many results will hold for both the tensor algebra $\mathcal{A}(X, \sigma)$ and the semicrossed product $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$. We will use $\mathcal{A}$ to denote either algebra, and will specify when the results diverge.

Characters. Let $\mathfrak{M}_\mathcal{A}$ denote the space of characters of $\mathcal{A}$ endowed with the weak-$*$ topology. Since $\mathcal{A}$ contains $\mathrm{C}_0(X)$ as a subalgebra, the restriction of any character θ to $\mathrm{C}_0(X)$ will be a point evaluation δ_x at some point $x \in X$. Let $\mathfrak{M}_{\mathcal{A},x}$ denote the set of all characters extending δ_x. Observe that since there is an expectation E of $\mathcal{A}$ onto $\mathrm{C}_0(X)$, there is always a distinguished character $\theta_{x,0} = \delta_x E$ in $\mathfrak{M}_{\mathcal{A},x}$. Since characters are always continuous, a character $\theta \in \mathfrak{M}_{\mathcal{A},x}$ is determined by $z = (\theta(\mathfrak{s}_1), \dots, \theta(\mathfrak{s}_n))$. We will write $\theta_{x,z}$ for this character when it is defined.

Lemma 7.1. *Let $x \in X$, and let $I_x = \{i : 1 \le i \le n,\ \sigma_i(x) = x\}$. Then*

$$\mathfrak{M}_{\mathcal{A}(X,\sigma),x} = \{\theta_{x,z} : z_i = 0 \text{ for } i \notin I_x,\ \|z\|_2 \le 1\} =: \overline{\mathbb{B}(I_x)}$$

and

$$\mathfrak{M}_{\mathrm{C}_0(X)\times_\sigma \mathbb{F}_n^+,x} = \{\theta_{x,z} : z_i = 0 \text{ for } i \notin I_x,\ \|z\|_\infty \le 1\} =: \overline{\mathbb{D}(I_x)}.$$

Moreover for each $a \in \mathcal{A}$, the function $\Theta_a(z) = \theta_{x,z}(a)$ is analytic on the ball (respectively polydisc) of radius 1 in the variables $\{z_i : i \in I_x\}$ and is continuous on the closure.

In particular, $\mathfrak{M}_{\mathcal{A},x} = \{\theta_{x,0}\}$ if x is not a fixed point for any σ_i.

Proof. Let $\theta \in \mathfrak{M}_{\mathcal{A},x}$. Characters always have norm 1; and indeed, they are completely contractive. So for $\mathcal{A}(X,\sigma)$, we must have

$$\|z\|_2 = \|\theta^{(1,n)}(\mathfrak{s})\| \le 1.$$

In case of the semicrossed product, we obtain $\|z\|_\infty \le 1$.

If $\sigma_i(x) = y \ne x$, then select a function $f \in \mathrm{C}_0(X)$ such that $f(x) = 0$ and $f(y) = 1$. Then

$$\begin{aligned} 0 &= f(x)\theta(\mathfrak{s}_i) = \theta(f)\theta(\mathfrak{s}_i) = \theta(f\mathfrak{s}_i) \\ &= \theta(\mathfrak{s}_i(f\circ\sigma_i)) = \theta(\mathfrak{s}_i)f(y) = \theta(\mathfrak{s}_i). \end{aligned}$$

So now suppose that $z \in \overline{\mathbb{B}(I_x)}$. Define a one-dimensional representation of $\mathcal{A}(X,\sigma)$ by setting $\theta(f) = f(x)$ and $\theta(\mathfrak{s}_i) = z_i$. Then the fact that $\sigma_i(x) = x$ for $i \in I_x$ ensures that the covariance relations are satisfied. Since $\|z\|_2 \le 1$, this is a row contractive representation. Hence this extends to a contractive representation of $\mathcal{A}(X,\sigma)$, yielding the desired character $\theta_{x,z}$. Similarly if $z \in \overline{\mathbb{D}(I_x)}$, then this determines a contractive covariant representation. So it extends to a character of $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$.

Now consider analyticity. If $a \in \mathcal{A}_0(X,\sigma)$, then $\Theta_a(z)$ is a polynomial in $\{z_i : i \in I_x\}$; and hence is analytic. Since $\|\Theta_a\|_\infty \le \|a\|$, it now follows for arbitrary a by approximation that Θ_a is the uniform limit of polynomials on $\overline{\mathbb{B}(I_x)}$ (respectively $\overline{\mathbb{D}(I_x)}$). Hence it is analytic on the interior of the ball (respectively polydisc) and continuous on the closure. ∎

Definition 7.2. An open subset M of $\mathfrak{M}_\mathcal{A}$ is called *analytic* if there is a domain Ω in $\mathbb{C}^d$ and a continuous bijection Θ of Ω onto M so that the function $\Theta(z)(a)$ is analytic on Ω for every $a \in \mathcal{A}$. It is a *maximal analytic subset* if it is maximal among analytic subsets of $\mathfrak{M}_\mathcal{A}$.

In particular, consider $x \in X$ for which I_x is non-empty. The open ball $\mathbb{B}(I_x) = \{z \in \overline{\mathbb{B}(I_x)} : \|z\|_2 < 1\}$ considered as an open subset of $\mathbb{C}^{|I_x|}$ is a complex domain that is mapped in the obvious way onto $B_x = \{\theta_{x,z} : z \in \mathbb{B}(I_x)\}$. This is an analytic set in $\mathfrak{M}_{\mathcal{A}(X,\sigma)}$. Similarly the polydisc $\mathbb{D}(I_x) = \{z \in \overline{\mathbb{D}(I_x)} : \|z\|_\infty < 1\}$ maps onto the analytic set $D_x = \{\theta_{x,z} : z \in \mathbb{D}(I_x)\}$ in $\mathfrak{M}_{\mathrm{C}_0(X)\times_\sigma \mathbb{F}_n^+}$.

Lemma 7.3. *The maximal analytic sets in $\mathfrak{M}_{A(X,\sigma)}$ are precisely the balls B_x for those points $x \in X$ fixed by at least one σ_i. Similarly the maximal analytic sets in $\mathfrak{M}_{\mathrm{C}_0(X)\times_\sigma \mathbb{F}_n^+}$ are precisely the polydiscs D_x for those points $x \in X$ fixed by at least one σ_i.*

Proof. It suffices to show that analytic sets must sit inside one of the fibres $\mathfrak{M}_{\mathcal{A},x}$. Let Θ map a domain Ω into $\mathfrak{M}_{\mathcal{A}}$. For every $f \in \mathrm{C}_0(X)$, the function $\Theta(z)(f)$ and the function $\Theta(z)(\overline{f}) = \overline{\Theta(z)(f)}$ are analytic, and hence constant. Since continuous functions on X separate points, this implies that Θ maps into a single fibre.

Now observe that $\mathfrak{M}_{\mathcal{A},x}$ is homeomorphic to a closed ball (respectively polydisc). Hence the maximal analytic subset would be the "interior" B_x (respectively D_x). So we have identified all of the maximal analytic sets. ∎

Corollary 7.4. *The characters of $\mathcal{A}$ determine X up to homeomorphism, and identify which points are fixed by some σ_i's, and determine exactly how many of the maps fix the point.*

Proof. The lemma shows that $\mathfrak{M}_{\mathcal{A}}$ consists of a space which is fibred over X, and the fibres are determined canonically as the closures of maximal analytic sets and the remaining singletons. Thus there is a canonical quotient map of $\mathfrak{M}_{\mathcal{A}}$ onto X; and this determines X. Next, the points which are fixed by some σ_i are exactly the points with a non-trivial fibre of characters. The corresponding maximal analytic set is homeomorphic to a ball (respectively polydisc) in $\mathbb{C}^d$ where $d = |I_x|$. The invariance of domain theorem shows that the dimension d is determined by the topology. ∎

Corollary 7.5. *If (X,σ) has a point fixed by two or more of the maps σ_i, then $\mathcal{A}(X,\sigma)$ and $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ are not algebraically isomorphic.*

Proof. An algebra isomorphism will yield a homeomorphism of the character spaces, and will be a biholomorphic map of each maximal analytic set of the tensor algebra to the corresponding maximal analytic set in the semicrossed product. The existence of a point x_0 fixed by $k \geq 2$ of the maps σ_i means that the tensor algebra contains the ball $\mathbb{B}_k$ as a maximal analytic set, while the semicrossed product has a polydisc $\mathbb{D}_k$. However no polydisk of dimension at least 2 is biholomorphic to any ball, and vice versa. Therefore the algebras are not isomorphic. ∎

Nest representations. In [**9**], we considered representations *onto* the 2×2 upper triangular matrices $\mathfrak{T}_2$. Here we actually need to consider a more general notion.

Definition 7.6. Let $\mathcal{N}_2$ denote the maximal nest $\big\{\{0\}, \mathbb{C}e_1, \mathbb{C}^2\big\}$ in $\mathbb{C}^2$. If $\mathcal{A}$ is an operator algebra, let $\operatorname{rep}_{\mathcal{N}_2}$ denote the collection of all continuous representations ρ of $\mathcal{A}$ on $\mathbb{C}^2$ such that $\operatorname{Lat}\rho(\mathcal{A}) = \mathcal{N}_2$.

These representations are called *nest representations*. There are two unital subalgebras of the 2×2 matrices $\mathfrak{M}_2$ with this lattice of invariant subspaces, $\mathfrak{T}_2$ and the abelian algebra $\mathcal{A}(E_{12}) = \operatorname{span}\{I, E_{12}\}$. Both have non-trivial radical. The other unital subalgebras of $\mathfrak{T}_2$ are semisimple, and except for $\mathbb{C}I$, they are all similar to the diagonal algebra $\mathfrak{D}_2$. Their lattice of invariant subspaces is the Boolean algebra with two generators. So representations with semisimple range are not nest representations.

Observe that for any representation ρ of $\mathcal{A}$ into $\mathfrak{T}_2$, the compression to a diagonal entry is a homomorphism. Thus ρ determines two characters which we denote by $\theta_{\rho,1}$ and $\theta_{\rho,2}$. The map ψ taking a to the $1,2$-entry of $\rho(a)$ is a point derivation satisfying

$$\psi(ab) = \theta_{\rho,1}(a)\psi(b) + \psi(a)\theta_{\rho,2}(b) \quad \text{for} \quad a, b \in \mathcal{A}.$$

Define $\operatorname{rep}_{y,x}(\mathcal{A})$ to be those nest representations for which $\theta_{\rho,1} \in \mathfrak{M}_{\mathcal{A},y}$ and $\theta_{\rho,2} \in \mathfrak{M}_{\mathcal{A},x}$.

It is convenient to consider representations which restrict to $*$-representations on $\mathrm{C}_0(X)$. For (completely) contractive representations of $\mathrm{C}_0(X)$, this is automatic. It is also the case that representations of $\mathrm{C}_0(X)$ into $\mathfrak{M}_2$ are automatically continuous, and thus are diagonalizable. Here we need a stronger version of this fact. Let $\operatorname{rep}^d \mathcal{A}$ and $\operatorname{rep}^d_{y,x} \mathcal{A}$ denote the nest representations which are diagonal on $\mathrm{C}_0(X)$.

Lemma 7.7. *Let X be a locally compact space; and let σ be a continuous map of X into itself. Let $K \subset X$, and let Ω be a domain in $\mathbb{C}^d$. Suppose that ρ is a map from $K \times \Omega$ into* $\operatorname{rep}\mathcal{A}$ *so that*

1. $\rho_{x,z} := \rho(x,z) \in \operatorname{rep}_{x,\sigma(x)} \mathcal{A}$ *for $x \in K$ and $z \in \Omega$.*
2. *ρ is continuous in the point–norm topology.*
3. *For each fixed $x \in K$, $\rho(x,z)$ is analytic in $z \in \Omega$.*

Then there exists a map A of $K \times \Omega$ into $\mathfrak{T}_2^{-1}$, the group of invertible upper triangular matrices, so that

1. $A(x,z)\rho_{x,z}(\cdot)A(x,z)^{-1} \in \operatorname{rep}^d \mathcal{A}$.
2. *$A(x,z)$ is continuous on $\big(K \setminus \{x : \sigma(x) = x\}\big) \times \Omega$.*
3. *For each fixed $x \in K$, $A(x,z)$ is analytic in $z \in \Omega$.*
4. $\max\{\|A(x,z)\|, \|A(x,z)^{-1}\|\} \leq 1 + \|\rho_{x,z}\|$.

Proof. When $\sigma(x) = x$, every representation $\rho \in \operatorname{rep}_{x,x} \mathcal{A}$ satisfies $\rho(f) = f(x)I_2$, and so they are automatically diagonalized. Define $A(x,z) = I_2$. This clearly satisfies conclusions (1), (3) and (4).

Now consider a point x with $\sigma(x) \neq x$. Choose a compact neighbourhood $\overline{\mathcal{V}}$ of x so that $\sigma(\overline{\mathcal{V}})$ is disjoint from $\overline{\mathcal{V}}$. Select $f \in \mathrm{C}_0(X)$ so that $f(\sigma(\overline{\mathcal{V}})) = \{1\}$ and $f(\overline{\mathcal{V}}) = \{0\}$. Let F be the function on $\overline{\mathcal{V}} \times \Omega$ given by the $1,2$ entry of $\rho_{x,z}(f)$; i.e.

$$\rho_{x,z}(f) = \begin{bmatrix} 1 & F(x,z) \\ 0 & 0 \end{bmatrix}.$$

Clearly F is continuous, and is analytic in z for each point in $\overline{\mathcal{V}}$.

The function F does not depend on the choice of f; for if g is another such function, the difference $f - g$ vanishes on $\overline{\mathcal{V}} \cup \sigma(\overline{\mathcal{V}})$. It is routine to factor this as the product $h_1 h_2$ of two functions vanishing on this set. But then

$$\rho_{x,z}(f-g) = \rho_{x,z}(h_1)\rho_{x,z}(h_2)$$

is the product of two strictly upper triangular matrices, and hence is 0. In particular, we may suppose that $\|f\| = 1$.

We can now define a function on $\big(K \setminus \{x : \sigma(x) = x\}\big) \times \Omega$ by

$$A(x,z) := \begin{bmatrix} 1 & F(x,z) \\ 0 & 1 \end{bmatrix}.$$

This is well defined by the previous paragraph. It is continuous, and is analytic in the second variable, because it inherits this from F.

For a point x with $\sigma(x) \neq x$, use the function f selected above and compute

$$\begin{aligned} A(x,z)\rho_{x,z}(f)A(x,z)^{-1} &= \begin{bmatrix} 1 & F(x,z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & F(x,z) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -F(x,z) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Similarly, if $h \in \mathrm{C}_0(X)$ and $h(\overline{\mathcal{V}} \cup \sigma(\overline{\mathcal{V}})) = \{1\}$, then

$$A(x,z)\rho_{x,z}(h)A(x,z)^{-1} = I_2.$$

Since the image of $\mathrm{C}_0(X)$ under any representation in $\operatorname{rep}_{x,\sigma(x)} \mathcal{A}$ is at most two dimensional, we conclude that $A(x,z)\rho_{x,z}(.)A(x,z)^{-1}$ is diagonal when restricted to $\mathrm{C}_0(X)$.

Finally observe that

$$\|A(x,z)\| = \|A(x,z)^{-1}\| \leq 1 + \|\rho_{x,z}\|\|f\| = 1 + \|\rho_{x,z}\|. \qquad \blacksquare$$

Example 7.8. For each $x \in X$ and $1 \le i \le n$, we define a nest representation $\rho_{x,i}$ by

$$\rho_{x,i}(f) = \begin{bmatrix} f(\sigma_i(x)) & 0 \\ 0 & f(x) \end{bmatrix} \quad \text{and} \quad \rho_{x,i}(\mathfrak{s}_j) = \begin{bmatrix} 0 & \delta_{ij} \\ 0 & 0 \end{bmatrix}$$

where δ_{ij} is the Kronecker delta function. Thus

$$\rho_{x,i}\big(\sum_{w \in \mathbb{F}_n^+} \mathfrak{s}_w f_w \big) = \begin{bmatrix} f_\varnothing(\sigma_i(x)) & f_i(x) \\ 0 & f_\varnothing(x) \end{bmatrix}$$

This has $\theta_{\rho_{x,i},1} = \theta_{\sigma_i(x),0}$ and $\theta_{\rho_{x,i},2} = \theta_{x,0}$. The 1, 2-entry $\psi_{\rho_{x,i}}(a) = f_i(x)$ is easily see to satisfy the derivation condition. This yields a nest representation of $\mathcal{A}$. Since one generator is sent to a contraction and the rest are sent to 0, this clearly defines a completely contractive representation of both $\mathcal{A}(X,\sigma)$ and $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$.

This representation maps $\mathcal{A}$ onto $\mathfrak{T}_2$ if $\sigma_i(x) \ne x$, and onto $\mathcal{A}(E_{12})$ when $\sigma_i(x) = x$.

The key to recovering the system (X,σ) from $\mathcal{A}$ is the following lemma.

Lemma 7.9. *If* $\mathrm{rep}_{y,x}(\mathcal{A})$ *is non-empty, then there is some* i *so that* $\sigma_i(x) = y$. *Furthermore, if* $\rho \in \mathrm{rep}^d_{y,x}(\mathcal{A})$ *and* $\sigma_j(x) \ne y$, *then* $\rho(\mathfrak{s}_j g)$ *is diagonal for all* $g \in \mathrm{C}_0(X)$.

Proof. Let $\rho \in \mathrm{rep}_{y,x}(\mathcal{A})$. By Lemma 7.7, we may assume that ρ belongs to $\mathrm{rep}^d_{y,x}(\mathcal{A})$. So for $f \in \mathrm{C}_0(X)$, we have $\rho(f) = \left[\begin{smallmatrix} f(y) & 0 \\ 0 & f(x) \end{smallmatrix}\right]$. If $\sigma_j(x) \ne y$, choose a function $f \in \mathrm{C}_0(X)$ so that $f(y) = 1$ and $f(\sigma_j(x)) = 0$. Let $g \in \mathrm{C}_0(X)$ and s be the 1, 2 entry of $\rho(\mathfrak{s}_j g)$. Apply ρ to the identity $f\mathfrak{s}_j g = \mathfrak{s}_j g(f \circ \sigma_j)$ to obtain

$$\begin{aligned} \begin{bmatrix} * & s \\ 0 & * \end{bmatrix} &= \begin{bmatrix} f(y) & 0 \\ 0 & f(x) \end{bmatrix} \begin{bmatrix} * & s \\ 0 & * \end{bmatrix} \\ &= \begin{bmatrix} * & s \\ 0 & * \end{bmatrix} \begin{bmatrix} f(\sigma_i(y)) & 0 \\ 0 & f(\sigma_i(x)) \end{bmatrix} = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Hence $s = 0$, and so $\rho(\mathfrak{s}_j g)$ is diagonal for all $g \in \mathrm{C}_0(X)$.

If $\rho(\mathfrak{s}_i g)$, $g \in \mathrm{C}_0(X)$, are all diagonal for $1 \le i \le n$, then $\rho(\mathcal{A})$ is diagonal—and thus not a nest representation. So there is some i for which $\rho(\mathfrak{s}_i g)$ is not diagonal; and thus $\sigma_i(x) = y$. ∎

8. Piecewise conjugate multisystems

In this section we introduce the concept of piecewise conjugacy for multivariable dynamical systems. This concept is central for the classification of their algebras and apparently new in the theory of dynamical systems. We therefore study some of its basic properties and present a few illuminating examples.

Recall that two multivariable dynamical systems (X, σ) and (Y, τ) are said to be *conjugate* if there exists a homeomorphism γ of X onto Y and a permutation $\alpha \in S_n$ so that $\gamma^{-1}\tau_i\gamma = \sigma_{\alpha(i)}$ for $1 \le i \le n$.

Definition 8.1. Say that two multivariable dynamical systems (X, σ) and (Y, τ) are *piecewise conjugate* if there is a homeomorphism γ of X onto Y and an open cover $\{\mathcal{U}_\alpha : \alpha \in S_n\}$ of X so that for each $\alpha \in S_n$,

$$\gamma^{-1}\tau_i\gamma|_{\mathcal{U}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{U}_\alpha}.$$

We could have expressed this somewhat differently, saying that Y has an open cover so that on each open set, there is some permutation α so that γ locally intertwines each τ_i with $\sigma_{\alpha(i)}$. But this is readily seen to be equivalent.

The difference in the two concepts of conjugacy lies on the fact that the permutations depend on the particular open set. As we shall see, a single permutation generally will not suffice.

For a point $x \in X$, we say that two continuous functions f, g mapping X into another topological space Z are equivalent if they agree on an open neighbourhood of x. The equivalence class $[f]_x$ is called the *germ* of f at x.

Let $\sigma(x) = \{\sigma_1(x), \dots .\sigma_n(x)\}$.

Proposition 8.2. *Let (X, σ) and (Y, τ) be piecewise conjugate multivariable dynamical systems. Assume that X is connected and that $E := \{x \in X : |\sigma(x)| = n\}$ is dense in X. Then (X, σ) and (Y, τ) are conjugate.*

Proof. Let γ be the homeomorphism implementing the piecewise conjugacy. Observe that by continuity, E is open; and by hypothesis, it is dense. For each $\alpha \in S_n$, define an open set

$$E_\alpha = \{x \in X : [\gamma^{-1}\tau_i\gamma]_x = [\sigma_{\alpha(i)}]_x \text{ for } 1 \le i \le n\}.$$

From the piecewise conjugacy, this is an open cover of X. If $x \in E$, the permutation α must be unique since $\sigma(x)$ consists of n distinct points. This must persist to the closure, for if $E_\alpha \cap E_\beta$ is non-empty, then being open, it would intersect E, contrary to the previous conclusion. Hence the sets E_α must be clopen. From the connectedness of X, we conclude that there is a single permutation α so that $X = E_\alpha$. This yields the desired conjugacy. ∎

Let $\mathcal{Z}(X,\sigma) = \{x \in X : |\sigma(x)| < n\}$. Then $\mathcal{Z}(X,\sigma)$ is the closed set E^c. Thus the previous proposition could be stated as saying that $\mathcal{Z}(X,\sigma)$ is nowhere dense.

If $\mathcal{Z}(X,\sigma)$ has non-empty interior, then the situation may change dramatically. We illustrate this in the simplest possible case.

Example 8.3. Let $X = [0,1]$ and $\sigma = (\sigma_1, \sigma_2)$. We describe the piecewise conjugacy class of $([0,1],\sigma)$. We may as well assume that $Y = X$ and $\gamma = \mathrm{id}$.

Pick points a_k, b_k, $k \in K$, so that the interior of $\mathcal{Z}(X,\sigma)$ can be expressed as a *disjoint* union

$$\mathcal{Z}(X,\sigma)^\circ = \bigcup_{k\in K} (a_k, b_k)$$

of (relatively) open subintervals of $[0,1]$. Select finitely many indices $k_1, k_2, \dots, k_l$. Without loss of generality, assume that $b_{k_j} < a_{k_{j+1}}$ for all $j = 1, 2, \dots, l-1$. For convenience, set $a_0 = b_0 = 0$ and $a_{l+1} = b_{l+1} = 1$. Define

$$\tau_1(x) = \begin{cases} \sigma_1(x) & \text{if} \quad a_{2j} \le x \le b_{2j+1} \\ \sigma_2(x) & \text{if} \quad a_{2j-1} \le x \le b_{2j} \end{cases}$$

and

$$\tau_2(x) = \begin{cases} \sigma_2(x) & \text{if} \quad a_{2j} \le x \le b_{2j+1} \\ \sigma_1(x) & \text{if} \quad a_{2j-1} \le x \le b_{2j} \end{cases}$$

Then $\mathcal{U}_{\mathrm{id}} = \bigcup (a_{2j}, b_{2j+1})$ and $\mathcal{U}_{(12)} = \bigcup (a_{2j-1}, b_{2j})$ is the partition for the piecewise conjugacy.

Conversely, any system $([0,1],\tau)$ which is piecewise conjugate to $([0,1],\sigma)$ arises that way. The main point is that the functions must coincide on an interval in order to make the switchover. On the other hand, there cannot be countably many switches, because the switching points would then have a cluster point, and the two systems could not coincide on any neighbourhood of that point. We omit the details.

In particular, suppose that σ_1 is non-increasing, σ_2 is non-decreasing and that $\mathcal{Z}([0,1],\sigma) = [a,b]$, with $0 < a < b < 1$. The switching across (a,b) yields a piecewise conjugate system (X,τ) which is not monotonic. As any homeomorphism of $[0,1]$ is monotone, these two systems are definitely not conjugate.

For $n = 2$, we can be more definitive in Proposition 8.2.

Proposition 8.4. *Let X be connected and let $\sigma = (\sigma_1, \sigma_2)$; and let $E := \{x \in X : |\sigma(x)| = 2\}$. Then piecewise conjugacy coincides with conjugacy if and only if $\overline{E}$ is connected.*

Proof. If E is empty, then $\sigma_1 = \sigma_2$ and there is nothing to prove. As in the proof of Proposition 8.2, the sets E_α are open and

$$E_{\rm id} \cap E_{(12)} \cap \overline{E} = \varnothing.$$

Arguing as before using the connectedness of $\overline{E}$, it follows that there is some α so that E_α contains $\overline{E}$. But clearly both $E_{\rm id}$ and $E_{(12)}$ contain $\overline{E}^c = \mathcal{Z}(X, \sigma)^\circ$. Thus $E_\alpha = X$ and so X and Y are conjugate. ∎

At the other end of the spectrum, total disconnectedness makes piecewise conjugacy very tractable.

Proposition 8.5. *Let X be a totally disconnected compact Hausdorff space. Fix a homeomorphism γ of X onto another space Y. Then there is a piecewise topological conjugacy of (X, σ) and (Y, τ) implemented by γ if and only if there is a partition of X into clopen sets $\{\mathcal{V}_\alpha : \alpha \in S_n\}$ so that for each $\alpha \in S_n$,*

$$\gamma^{-1} \tau_i \gamma|_{\mathcal{V}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{V}_\alpha}.$$

Proof. If (X, σ) and (Y, τ) are piecewise conjugate, let

$$\mathcal{U}_\alpha = \{x \in X : [\gamma^{-1}\tau_i\gamma]_x = [\sigma_{\alpha(i)}]_x,\ i \le i \le n\}.$$

By hypothesis, this is an open cover of X. Hence there is a partition of unity $\{f_\alpha\}$ of positive functions in $\mathrm{C}(X)$ such that $\sum f_\alpha = 1$ and $f_\alpha^{-1}(0, \infty) \subset \mathcal{U}_\alpha$ for $\alpha \in S_n$. Then $K_\alpha = f_\alpha^{-1}[1/n!, 1]$ is a compact subset of $\mathcal{U}_\alpha$ and $\bigcup_\alpha K_\alpha = X$.

Each point $x \in K_\alpha$ is contained in $\mathcal{U}_\alpha$, and thus there is a clopen neighbourhood V_x of x contained in $\mathcal{U}_\alpha$. Select a finite subcover of K_α. The union of this finite cover is a clopen set V_α containing K_α and contained in U_α.

To obtain the desired partition, order the permutations, and replace V_α by its intersection with the complement of the V_β's which precede it in the list. ∎

9. Classification

The purpose of this section is to establish the connection between isomorphism of the tensor or semicrossed product algebras and piecewise conjugacy.

Before embarking on the proof, we need to recall some basic facts from the theory of several complex variables [**15**]. Let $G \subseteq \mathbb{C}^k$ be a domain and let $A \subseteq G$ be an analytic variety of G. One says that A is regular at $z \in A$ of dimension q if there is a neighborhood $U_z \subseteq G$ of z and holomorphic functions $f_i : U_z \to \mathbb{C}$ for $1 \le i \le k-q$ so that

(i) the Jacobian of $(f_1, f_2, \dots, f_{k-q})$ at z has rank $k-q$, and,
(ii) $A \cap U_z = N(f_1, f_2, \dots f_{k-q}) := \bigcap_{i=1}^{k-q} f_i^{-1}(0)$.

By the Theorem of local parameterization at regular points [**15**, Theorem I.8.3], the dimension is well defined. The set A_{reg} of all regular points of A forms an open subset of A, and the dimension is constant on each connected component of A_{reg}. The closure of each such component is itself an analytic variety of G which is irreducible, i.e., the regular points are connected. The maximum dimension of the irreducible subvarieties of A is called the dimension of A.

We need a rather modest conclusion from this theory. However we know of no elementary proof of this fact.

Proposition 9.1. *Let $G \subseteq \mathbb{C}^k$ be a domain and let*

$$\psi = (\psi_1, \psi_2, \dots, \psi_l) : \mathbb{C}^k \to \mathbb{C}^l, \text{ where } l < k,$$

be a holomorphic function. Then the zero set $N(\psi_1, \psi_2, \dots, \psi_l)$ of ψ is either empty or infinite. In particular, the zero set has no isolated points.

Proof. Assume that $N(\psi_1, \psi_2, \dots, \psi_l)$ is non-empty. If $l = 1$, then the techniques of [**15**, Proposition I.8.4] imply that the dimension of that variety is $k-1$. Consequently, [**15**, Exercise III.6.1] implies that for an arbitrary k, the dimension of $N(\psi_1, \psi_2, \dots, \psi_l)$ is at least $k-l$ and hence is positive. In particular, some irreducible component of $N(\psi_1, \psi_2, \dots, \psi_l)$ is regular of positive dimension at some point z. The conclusion follows now from the Implicit Function Theorem.

If there were an isolated point of the zero set, simply reduce G to a small neighbourhood of that point to obtain a contradiction. ∎

We now have all the requisite tools to recover much of the dynamical system from the tensor or crossed product algebra in the multivariable setting. Again we will work as much as possible with both the tensor algebra and the semicrossed product.

Theorem 9.2. *Let (X, σ) and (Y, τ) be two multivariable dynamical systems. We write $\mathcal{A}$ and $\mathcal{B}$ to mean either $\mathcal{A}(X, \sigma)$ and $\mathcal{A}(Y, \tau)$ or $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ and $\mathrm{C}_0(Y) \times_\tau \mathbb{F}_n^+$. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as algebras, then the dynamical systems (X, σ) and (Y, τ) are piecewise conjugate.*

Proof. Let γ be an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. This induces a bijection γ_c from the character space $\mathfrak{M}_{\mathcal{A}}$ onto $\mathfrak{M}_{\mathcal{B}}$ by $\gamma_c(\theta) = \theta \circ \gamma^{-1}$. Similarly it induces a map γ_r from $\mathrm{rep}_{\mathcal{N}_2}(\mathcal{A})$ onto $\mathrm{rep}_{\mathcal{N}_2}(\mathcal{B})$.

Since $\mathfrak{M}_{\mathcal{A}}$ is endowed with the weak-$*$ topology, it is easy to see that γ_c is continuous. Indeed, if θ_α is a net in $\mathfrak{M}_{\mathcal{A}}$ converging to θ and $b \in \mathcal{B}$, then

$$\lim_\alpha \gamma_c\theta_\alpha(b) = \lim_\alpha \theta_\alpha(\gamma^{-1}(a)) = \theta(\gamma^{-1}(a)) = \gamma_c\theta(b).$$

The same holds for γ_c^{-1}. So γ_c is a homeomorphism.

Observe that γ_c carries analytic sets to analytic sets. Indeed, if Θ is an analytic function of a domain Ω into $\mathfrak{M}_{\mathcal{A}}$, then

$$\gamma_c\Theta(z)(b) = \Theta(z)(\gamma^{-1}(b))$$

is analytic for every $b \in \mathcal{B}$; and thus $\gamma_c\Theta$ is analytic. Since the same holds for γ^{-1}, it follows that γ_c takes maximal analytic sets to maximal analytic sets. Thus it carries their closures, $\mathfrak{M}_{\mathcal{A},x}$, onto sets $\mathfrak{M}_{\mathcal{B},y}$. The same also holds when these sets are singletons.

By Corollary 7.4, X is the quotient of $\mathfrak{M}_{\mathcal{A}}$ obtained by squashing each $\mathfrak{M}_{\mathcal{A},x}$ to a point. It follows that γ_c induces a set map γ_s of X onto Y which is a homeomorphism since both X and Y inherit the quotient topology.

Fix $x_0 \in X$, and let $y_0 = \gamma_c(x_0)$. Fix one of the maps σ_{i_0}, and consider the set

$$\mathcal{F} = \{\sigma_i, \gamma^{-1}\tau_j\gamma : [\sigma_i]_x = [\sigma_{i_0}]_x = [\gamma^{-1}\tau_j\gamma]_x\}.$$

For convenience, let us relabel so that $i_0 = 1$ and

$$\mathcal{F} = \{\sigma_1, \dots, \sigma_k, \gamma^{-1}\tau_1\gamma, \dots, \gamma^{-1}\tau_l\gamma\}.$$

Fix a neighbourhood $\mathcal{V}$ of x_0 on which all of these functions agree, and such that $\overline{\mathcal{V}}$ is compact. Furthermore, if $\sigma_1(x_0) \neq x_0$, then choose $\mathcal{V}$ so that $\overline{V} \cap \sigma_1(\overline{\mathcal{V}}) = \emptyset$.

The hard part of the proof is to show that $k = l$. Assume that this has been verified. Then one can partition the functions σ_i into families with a common germ at x_0, and they will be paired with a corresponding partition of the $\gamma^{-1}\tau_j\gamma$'s of equal size. This provides the desired permutation in some neighbourhood of each x_0.

It remains to show that $k = l$. By way of contradiction, assume that $k \neq l$. By exchanging the roles of X and Y if necessary, we may assume that $k > l$. We do not exclude the possibility that $l = 0$.

For any $x \in \mathcal{V}$ and $z = (z_1, z_2, \dots, z_l) \in \mathbb{C}^k$, consider the covariant representations $\rho_{x,z}$ of $\mathcal{A}_0(X, \sigma)$ into $\mathfrak{M}_2$ defined by

$$\rho_{x,z}(f) = \begin{bmatrix} f(\sigma_1(x)) & 0 \\ 0 & f(x) \end{bmatrix},$$

$$\rho_{x,z}(\mathfrak{s}_i) = \begin{bmatrix} 0 & z_i \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad 1 \leq i \leq k,$$

and

$$\rho_{x,z}(\mathfrak{s}_i) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad k < i \le n.$$

This extends to a well defined representation of $\mathcal{A}$, where a typical element $A \sim \sum_{w \in \mathbb{F}_n^+} \mathfrak{s}_w f_w$ is sent to

$$\rho_{x,z}(A) = \begin{bmatrix} f_0(\sigma_1(x)) & \sum_{i=1}^k f_i(x) z_i \\ 0 & f_0(x) \end{bmatrix}$$

There are no continuity problems since the Fourier coefficients are continuous.

This representation will be (completely) contractive if $z \in \overline{\mathbb{B}}_k$ for $\mathcal{A}(X, \sigma)$ and $z \in \overline{\mathbb{D}}_k$ for the semicrossed product. For other values of z, these representations are similar to completely contractive representations by conjugating by $\operatorname{diag}(\|z\|_2, 1)$ or $\operatorname{diag}(\|z\|_\infty, 1)$ respectively. Thus the norm can be estimated as $\|\rho_{x,z}\| \le \|z\|$, where we use the 2-norm or the max norm depending on whether we are considering the tensor algebra or the semicrossed product.

The representation $\rho_{x,z}$ maps into $\mathfrak{T}_2$ and is a nest representation in $\operatorname{rep}_{\sigma_1(x),x} \mathcal{A}$ when $z \ne 0$, but is diagonal at $z = 0$. Observe that the range of $\rho_{x,z}$ for $z \ne 0$ equals $\mathfrak{T}_2$ when $\sigma_1(x) \ne x$ and equals $\mathcal{A}(E_{12})$ when $\sigma_1(x) = x$. Moreover this map is point–norm continuous, and is analytic in the second variable.

Now consider the map defined on $\overline{\mathcal{V}} \times \mathbb{C}^k$ given by

$$\Phi_0(x, z) = \gamma_r(\rho_{x,z}) \in \operatorname{rep}_{\gamma_s \sigma_1(x), \gamma_s(x)} \mathcal{B}.$$

By Corollary 4.6 or Corollary 6.9, γ is continuous; and so γ_r is also continuous. Thus Φ_0 is point–norm continuous, and is analytic in the second variable. So Φ_0 fulfils the requirements of Lemma 7.7. Hence there exists a map $A(x, z)$ of $\overline{\mathcal{V}} \times \mathbb{C}^k$ into $\mathfrak{T}_2^{-1}$, which is analytic in the second variable, so that

$$\Phi(x, z) = A(x, z)\gamma_r(\rho_{x,z})A(x, z)^{-1}$$

diagonalizes $\mathrm{C}_0(Y)$. Moreover

$$\max\{\|A(x, z)\|, \|A(x, z)^{-1}\|\} \le 1 + \|\gamma_r\| \|z\|.$$

Recall that when $\sigma_1(x_0) \ne x_0$, we chose $\mathcal{V}$ so that $\overline{\mathcal{V}}$ is disjoint from $\sigma_1(\overline{\mathcal{V}})$. Therefore in this case, A is a continuous function.

Choose $h \in \mathrm{C}_0(Y)$ such that $h|_{\gamma_s(\overline{\mathcal{V}})} = 1$ and $\|h\|_\infty = 1$. Define $\psi_j(z)$ to be the $1, 2$ entry of $\Phi(x_0, z)(\mathfrak{t}_j h)$; and set $\Psi(z) = (\psi_1(z), \dots, \psi_n(z))$. Then Ψ is an analytic function from $\mathbb{C}^k$ into $\mathbb{C}^n$.

We claim that $\psi_j(z) = 0$ *for* $j > l$.

Indeed, since $j > l$, the map $\gamma^{-1}\tau_j\gamma$ is not in $\mathcal{F}$. Hence there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in $\overline{\mathcal{V}}$ converging to x_0 so that $\gamma_s^{-1}\tau_j\gamma_s(x_\lambda) \neq \sigma_1(x_\lambda)$ for all $\lambda \in \Lambda$. By Lemma 7.9, $\Phi(x_\lambda, z)(\mathfrak{t}_j h)$ is diagonal for all λ in Λ.

First consider the case when $\sigma_1(x_0) \neq x_0$. Then $A(x, z)$ is continuous, and so $\Phi(x, z)$ is point–norm continuous. Taking limits, we conclude that $\Phi(x_0, z)(\mathfrak{t}_j h)$ is diagonal; whence $\psi_j(z) = 0$.

Now consider the case $\sigma_1(x_0) = x_0$. Recall that in this case, $\Phi(x_0, z)$ has range in $\mathcal{A}(E_{12})$; so that the diagonal part consists of scalars. Fix $z \in \mathbb{C}^k$. Since

$$\max\{\|A(x_\lambda, z)\|, \|A(x_\lambda, z)^{-1}\|\} \leq 1 + \|\gamma_r\|\|z\|,$$

we may pass to a subnet if necessary so that $\lim_\Lambda A(x_\lambda, z) = A(z)$ exists in $\mathfrak{T}_2^{-1}$. Since Φ_0 is point–norm continuous and $A(x_0, z) = I_2$,

$$\begin{aligned}\lim_{\lambda\in\Lambda} \Phi(x_\lambda, z)(\mathfrak{t}_j h) &= \lim_{\lambda\in\Lambda} A(x_\lambda, z)\Phi_0(x_\lambda, z)(\mathfrak{t}_j h)A(x_\lambda, z)^{-1} \\ &= A(z)\Phi_0(x_0, z)(\mathfrak{t}_j h)A(z)^{-1}.\end{aligned}$$

Therefore $A(z)\Phi_0(x_0, z)(\mathfrak{t}_j h)A(z)^{-1}$ is diagonal, and hence scalar. So $\Phi(x_0, z)(t_j h)$ is scalar and $\psi_j(z) = 0$, which proves the claim.

The function Ψ can now be considered as an analytic function from $\mathbb{C}^k$ into $\mathbb{C}^l$. Observe that $\Psi(0) = 0$. By Proposition 9.1 there exists $z_0 \neq 0$ for which $\Psi(z_0) = 0$. Then $\Phi(z_0)$ is diagonal, and thus is not a nest representation. This is a contradiction which proves the theorem. ∎

In light of Proposition 8.2 we obtain the following classification for both the tensor algebra and the semicrossed product. Recall that $\mathcal{Z}(X, \sigma) = \{x : |\sigma(x)| < n\}$.

Corollary 9.3. *Let (X, σ) and (Y, τ) be multivariable dynamical systems and assume that X is connected and $\mathcal{Z}(X, \sigma)$ has empty interior. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if the systems (X, σ) and (Y, τ) are conjugate.*

Corollary 9.3 is the strongest result we can offer for both the tensor algebra and the semi-crossed product. In order to obtain more definitive results, we focus in the tensor algebra case. In the totally disconnected case, the situation is straightforward. The reader should compare this with [**19**].

Corollary 9.4. *Assume that X is totally disconnected. Then the tensor algebras $\mathcal{A}(X, \sigma)$ and $\mathcal{A}(Y, \tau)$ are isomorphic if and only if (X, σ) and (Y, τ) are piecewise topologically conjugate. In this case, the algebras are completely isometrically isomorphic.*

Proof. One direction is provided by Theorem 9.2. For the converse, assume that (X,σ) and (Y,τ) are piecewise topologically conjugate. By Proposition 8.5, there is a partition of X into clopen sets $\{\mathcal{V}_\alpha : \alpha \in S_n\}$ so that for each $\alpha \in S_n$,

$$\gamma^{-1}\tau_i\gamma|_{\mathcal{V}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{V}_\alpha}.$$

We may assume that $\mathcal{A}(X,\sigma)$ is contained in a universal operator algebra generated by $\mathrm{C}_0(X)$ and n isometries $\mathfrak{s}_1,\dots,\mathfrak{s}_n$ satisfying the covariance relations. Likewise, $\mathcal{A}(Y,\tau)$ is contained in the corresponding algebra generated by $\mathrm{C}_0(Y)$ and $\mathfrak{t}_1,\dots,\mathfrak{t}_n$. Define a covariant representation of (Y,τ) by

$$\begin{aligned}\varphi(f) &= f\circ\gamma\\ \varphi(\mathfrak{t}_i) &= \sum_{\alpha\in S_n}\mathfrak{s}_{\alpha(i)}\chi_{\mathcal{V}_\alpha} \quad\text{for}\quad 1\le i\le n.\end{aligned}$$

To see that $\big[\varphi(\mathfrak{t}_i)\ \dots\ \varphi(\mathfrak{t}_n)\big]$ is a row isometry, compute

$$\begin{aligned}\varphi(\mathfrak{t}_j)^*\varphi(\mathfrak{t}_i) &= \sum_{\alpha,\beta\in S_n}\chi_{\mathcal{V}_\beta}\mathfrak{s}^*_{\beta(j)}\mathfrak{s}_{\alpha(i)}\chi_{\mathcal{V}_\alpha}\\ &= \sum_{\alpha,\beta\in S_n}\delta_{\beta(j),\alpha(i)}\chi_{\mathcal{V}_\beta}\chi_{\mathcal{V}_\alpha}\\ &= \sum_{\alpha\in S_n}\delta_{\alpha(j),\alpha(i)}\chi_{\mathcal{V}_\alpha} = \delta_{j,i}I.\end{aligned}$$

So $\varphi(t_i)$ are isometries with pairwise orthogonal ranges.

Next observe that φ satisfies the covariance relations.

$$\begin{aligned}\varphi(f)\varphi(t_i) &= (f\circ\gamma)\sum_{\alpha\in S_n}\mathfrak{s}_{\alpha(i)}\chi_{\mathcal{V}_\alpha}\\ &= \sum_{\alpha\in S_n}\mathfrak{s}_{\alpha(i)}\chi_{\mathcal{V}_\alpha}(f\circ\gamma\sigma_{\alpha(i)})\end{aligned}$$

and since $\gamma\sigma_{\alpha(i)} = \tau_i\gamma$ on $\mathcal{V}_\alpha$,

$$= \sum_{\alpha\in S_n}\mathfrak{s}_{\alpha(i)}\chi_{\mathcal{V}_\alpha}(f\circ\tau_i\gamma) = \varphi(\mathfrak{t}_i)\varphi(f\circ\tau_i).$$

Therefore φ extends to a completely contractive representation of $\mathcal{A}(Y,\tau)$ given by

$$\varphi\Big(\sum_{w\in\mathbb{F}_n^+}\mathfrak{t}_w f_w\Big) = \sum_{w\in\mathbb{F}_n^+}\varphi(\mathfrak{t}_w)(f_w\circ\gamma)$$

where $w = i_k\dots i_1$ and $\varphi(\mathfrak{t}_w) = \varphi(\mathfrak{t}_{i_k})\dots\varphi(\mathfrak{t}_{i_1})$. It is evident that this maps $\mathcal{A}(Y,\tau)$ into $\mathcal{A}(X,\sigma)$.

However the relations between σ and τ can be reversed to obtain a completely contractive map from $\mathcal{A}(X,\sigma)$ into $\mathcal{A}(Y,\tau)$. A little thought shows that this map is the inverse of φ, verifying that φ is a completely isometric isomorphism. ∎

Let us define the graph of σ to be

$$G(\sigma) = \{(x, \sigma_i(x)) : x \in X,\ 1 \le i \le n\}$$

considered as a subset of $X \times X$.

Corollary 9.5. *If the maps σ_i have disjoint graphs for $1 \le i \le n$, then $\mathcal{A}(X,\sigma)$ is isomorphic to $\mathcal{A}(Y,\tau)$ if and only if there is a homeomorphism γ of X onto Y which implements a homeomorphism between the graphs of σ and τ. In this case, the algebras are completely isometrically isomorphic.*

Proof. By Theorem 9.2, an isomorphism between $\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ yields a piecewise topological conjugacy via a homeomorphism γ. In particular, this implements a homeomorphism from the graph $G(\sigma)$ onto $G(\tau)$ via the map $\gamma \times \gamma$.

Conversely suppose that γ is a homeomorphism of X onto Y so that $\gamma \times \gamma$ carries $G(\sigma)$ onto $G(\tau)$. For simplicity of notation, we may suppose that $Y = X$ and $\gamma = \mathrm{id}$. The fact that σ_i have disjoint graphs means that $|\sigma(x)| = n$ for each $x \in X$. Since $\tau(x) = \sigma(x)$, there is a unique permutation $\alpha_x \in S_n$ so that $\tau_i(x) = \sigma_{\alpha_x(i)}(x)$ for $1 \le i \le n$. A simple argument using the continuity of σ and τ shows that the map α taking $x \in X$ to $\alpha_x \in S_n$ is continuous. Therefore the sets $\mathcal{V}_\alpha = \{x : \alpha_x = \alpha\}$ yields a partition of X into clopen sets with the property that

$$\tau_i|_{\mathcal{V}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{V}_\alpha} \quad \text{for} \quad \alpha \in S_n \text{ and } 1 \le i \le n.$$

The proof is now completed as for Corollary 9.4. ∎

When $n = 2$, we have a complete answer.

Theorem 9.6. *Let $n = 2$ and suppose that (X, σ_1, σ_2) and (Y, τ_1, τ_2) are two dynamical systems. The following are equivalent:*

1. *(X,σ) and (Y,τ) are piecewise topologically conjugate.*
2. *$\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are isomorphic.*
3. *$\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are completely isometrically isomorphic.*
4. *There is a homeomorphism γ of X onto Y so that*
   (a) *$\{\gamma\sigma_1(x), \gamma\sigma_2(x)\} = \{\tau_1\gamma(x), \tau_2\gamma(x)\}$ for each $x \in X$, and*
   (b) *$X_i = \{x \in X : \gamma\sigma_1(x) \ne \tau_i\gamma(x)\}$, $i = 1, 2$, have disjoint closures.*

Proof. Clearly (3) implies (2); and (2) implies (1) by Theorem 9.2. Suppose that (1) holds for a homeomorphism γ and an open cover $\{\mathcal{U}_{\rm id}, \mathcal{U}_{(12)}\}$. That is, $\tau_i\gamma|_{\mathcal{U}_{\rm id}} = \gamma\sigma_i|_{\mathcal{U}_{\rm id}}$ and $\tau_i\gamma|_{\mathcal{U}_{(12)}} = \gamma\sigma_{i'}|_{\mathcal{U}_{(12)}}$ for $i = 1, 2$ and $\{i, i'\} = \{1, 2\}$. In particular, (4a) holds for every $x \in X$. Moreover, $\overline{X_1} \subset \mathcal{U}_{\rm id}^c$ and $\overline{X_2} \subset \mathcal{U}_{(12)}^c$. Consequently,

$$\overline{X_1} \cap \overline{X_2} \subset \big(\mathcal{U}_{\rm id} \cup \mathcal{U}_{(12)}\big)^c = \varnothing.$$

This establishes (4b).

So we now assume that (4) holds. To simplify the notation, we may assume that $Y = X$ and $\gamma = \mathrm{id}$. Let (π, S_1, S_2) be a faithful representation of the covariance relations for (X, σ). Since $\overline{X_1}$ and $\overline{X_2}$ are disjoint, there is a continuous function $h \in \mathrm{C}_b(X)$ such that $h|_{\overline{X_1}} = 0$, $h|_{\overline{X_2}} = \frac{\pi}{2}$ and h takes real values in $[0, \frac{\pi}{2}]$ everywhere. Let $\overline{\pi}$ denote the extension of π to the bounded Borel functions on X. Define

$$T_1 = S_1\overline{\pi}(\sin h) + S_2\overline{\pi}(\cos h) \quad \text{and} \quad T_2 = S_1\overline{\pi}(\cos h) - S_2\overline{\pi}(\sin h).$$

First observe that T_i are isometries with orthogonal range. For example,

$$T_i^*T_1 = \overline{\pi}(\sin h)S_1^*S_1\overline{\pi}(\sin h) + \overline{\pi}(\cos h)S_2^*S_2\overline{\pi}(\cos h) = I$$

and

$$T_1^*T_2 = \overline{\pi}(\sin h)S_1^*S_1\overline{\pi}(\cos h) - \overline{\pi}(\cos h)S_2^*S_2\overline{\pi}(\sin h) = 0.$$

Next verify the covariance relations $\pi(f)T_i = T_i\pi(f \circ \tau_i)$:

$$\begin{aligned} \pi(f)T_i &= \pi(f)S_1\overline{\pi}(\sin h) + \pi(f)S_2\overline{\pi}(\cos h) \\ &= S_1\overline{\pi}(\sin h)\pi(f \circ \sigma_1) + S_2\overline{\pi}(\cos h)\pi(f \circ \sigma_2) \end{aligned}$$

and since $\sigma_1 = \tau_1$ on $h^{-1}((0, \frac{\pi}{2}])$ and $\sigma_2 = \tau_1$ on $h^{-1}([0, \frac{\pi}{2}))$,

$$\begin{aligned} &= \big(S_1\overline{\pi}(\sin h) + S_2\overline{\pi}(\cos h)\big)\pi(f \circ \tau_1) \\ &= T_i\pi(f \circ \tau_i). \end{aligned}$$

Next observe that $\mathcal{A}(X, \sigma)$ is generated by $\mathrm{C}_0(X)$ and $T_i\mathrm{C}_0(X)$ for $i = 1, 2$. This is because

$$S_1 = T_1\overline{\pi}(\sin h) + T_2\overline{\pi}(\cos h) \quad \text{and} \quad S_2 = T_1\overline{\pi}(\cos h) - T_2\overline{\pi}(\sin h).$$

Multiplying on the right by $\pi(f)$ for any $f \in \mathrm{C}_0(X)$ yields the corresponding fact for $\mathcal{A}(X, \sigma)$.

Now exactly the same procedure works in a faithful representation of $\mathcal{A}(X, \tau)$. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ denote the generators of $\mathcal{A}(X, \tau)$. Since $\mathcal{A}(X, \sigma)$ contains a representation of the covariance relations for (X, τ), there is a homomorphism of $\mathcal{A}(X, \tau)$ onto $\mathcal{A}(X, \sigma)$ that takes f to $\pi(f)$ and $\mathfrak{t}_i f$ to $T_i f$. It is then clear that $a_1 := \mathfrak{t}_1\overline{\pi}(\sin h) + \mathfrak{t}_2\overline{\pi}(\cos h)$ is sent to

S_1 and $a_2 := \mathfrak{t}_1\overline{\pi}(\cos h) - \mathfrak{t}_2\overline{\pi}(\sin h)$ is sent to S_2. Likewise there is an algebra homomorphism in the reverse direction which is evidently the inverse map on the generators. Therefore these algebras are isomorphic. Moreover the maps in both directions are complete contractions, and thus are completely isometric. ∎

The evidence strongly suggests the following:

Conjecture 9.7. The tensor algebras $\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are isomorphic if and only if the systems (X,σ) and (Y,τ) are piecewise topologically conjugate; and in this case, they are completely isometrically isomorphic.

The analysis leads to a technical conjecture which we can verify in low dimensions that would suffice to solve our conjecture.

Conjecture 9.8. Let Π_n be the $n!$-simplex with vertices indexed by S_n. Is there a continuous function u of Π_n into $U(n)$ so that:

(1) each vertex is taken to the corresponding permutation matrix,

(2) for every pair of partitions (A,B) of the form
$$\{1,\dots,n\} = A_1\dot\cup\dots\dot\cup A_m = B_1\dot\cup\dots\dot\cup B_m,$$
where $|A_s| = |B_s|,\ 1 \le s \le m$, let
$$\mathcal{P}(A,B) = \{\alpha \in S_n : \alpha(A_s) = B_s, 1 \le s \le m\}.$$
If $x = \sum_{\alpha\in\mathcal{P}(A,B)} x_\alpha \alpha$, then the non-zero matrix coefficients of $u_{ij}(x)$ are supported on $\bigcup_{s=1}^m B_s \times A_s$. We call this the *block decomposition condition.*

Proposition 9.9. *Conjecture 9.8 is valid for* $n = 2,3$. *Moreover there is a function on the* 1*-skeleton of* Π_n *satisfying the conditions of this conjecture for every* n.

Proof. For each permutation $\alpha \in S_n$, one can choose a Hermitian matrix A_α so that $U_\alpha = \exp(iA_\alpha)$ as follows: Decompose α into cycles, and select a logarithm for each cycle with arguments in $[-\pi,\pi]$. The eigenvalues come in conjugate pairs except for ± 1. Make the choice between $\pm\pi$ for the eigenvalue -1 alternately; so that $\|A_\alpha\| \le \pi$ and $\operatorname{Tr} A_\alpha \in \{0,\pi\}$. Choosing A_α in this way ensures that it respects the block diagonal structure of U_α coming from the cycle decomposition.

To describe such a function on the 1-skeleton of Π_n, just order the elements of S_n. For future use, we insist that the even permutations precede the odd permutations in this order. Then if $\alpha < \beta$ in this order, define
$$u((1-t)\alpha + t\beta) = U_\alpha \exp(itA_{\alpha^{-1}\beta}).$$

This works because if $\alpha, \beta \in P(A, B)$, then $\alpha^{-1}\beta \in P(A, A)$. Hence $A_{\alpha^{-1}\beta}$ respects the block structure A, so that $\exp(itA_{\alpha^{-1}\beta})$ does also. Consequently $U_\alpha \exp(itA_{\alpha^{-1}\beta})$ respects the A, B block decomposition. This 1-skeleton argument contains the $n = 2$ case.

Observe that for any n, the function $u(\sum t_\alpha \alpha) = \exp(i \sum t_\alpha A_\alpha)$ maps Π_n into $U(n)$ and $u(\alpha) = U_\alpha$. n general, it does not satisfy the block decomposition condition.

Now consider $n = 3$. Any non-trivial block decomposition of 3×3 is given by a vertex (i, j); namely

$$\{A_1 = \{i\},\, A_2 = A_1^c,\, B_1 = \{j\},\, B_2 = B_1^c\}.$$

There are two permutations respecting this block decomposition, say α_{ij} and β_{ij} where α_{ij} is even and β_{ij} is odd. To find the function u satisfying Conjecture 9.8, we start with the function constructed in the previous paragraph and modify it on the nine edges to obtain the block decomposition condition.

Take the 6-simplex Π_3 and glue a half-disk D_{ij} to each of the nine edges$[\alpha_{ij}, \beta_{ij}]$. On the semicircular edge $\Gamma_{ij}[0, 1]$ between α_{ij} and β_{ij}, define u as in the second paragraph:

$$u(\Gamma_{ij}(t)) = U_{\alpha_{ij}} \exp(itA_{\alpha_{ij}^{-1}\beta_{ij}}).$$

Recall that on the straight edge $[\alpha_{ij}, \beta_{ij}]$, u is define as

$$u((1-t)\alpha_{ij} + t\beta_{ij}) = \exp(i(1-t)A_{\alpha_{ij}} + itA_{\beta_{ij}}).$$

The determinant along these two paths is respectively

$$\det(U_{\alpha_{ij}}) \exp(it \operatorname{Tr} A_{\alpha_{ij}^{-1}\beta_{ij}}) = e_{it\pi}$$

and

$$\exp(i(1-t) \operatorname{Tr} A_{\alpha_{ij}} + it \operatorname{Tr} A_{\beta_{ij}}) = e_{it\pi}$$

because $\operatorname{Tr} A_{ij} = 0$ and $\operatorname{Tr} A_{\beta_{ij}} = \operatorname{Tr} A_{\alpha_{ij}^{-1}\beta_{ij}} = \pi$.

The issue is now to extend the definition of u to each half-disk D_{ij} in a continuous way. The key fact is that $SU(3)$ is simply connected [**27**, Theorem II.4.12]. (Indeed $SU(n)$ is simply connected for all $n \geq 1$.) The determinant is an obstruction to $U(n)$ being simply connected. But by ensuring that our functions have domain in the set of unitaries with determinant in the upper half plane means that we remain in a simply connected set. Hence there is a homotopy between the edge $u([\alpha_{ij}, \beta_{ij}])$ and $u(\Gamma_{ij})$ which enables the extension of the definition of u. Finally observe that there is a continuous function h of Π_3 onto $\Pi_3 \bigcup_{i,j=1}^3 D_{ij}$ which takes the edges $[\alpha_{ij}, \beta_{ij}]$ onto Γ_{ij}. The composition $v = u \circ h$ is the desired function. ∎

We can now show how to modify the proof of Theorem 9.6 to work for any n for which we have Conjecture 9.8. Recall that the covering dimension of a topological space X is the smallest integer k so that every open cover can be refined so that each point is covered by at most $k+1$ points.

Theorem 9.10. *If Conjecture 9.8 is correct for some value of n, then for two paracompact dynamical systems (X,σ) and (Y,τ) where $\sigma = \{\sigma_1,\dots,\sigma_n\}$ and $\tau = \{\tau_1,\dots,\tau_n\}$, the following are equivalent:*
1. *(X,σ) and (Y,τ) are piecewise topologically conjugate.*
2. *$\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are isomorphic.*
3. *$\mathcal{A}(X,\sigma)$ and $\mathcal{A}(Y,\tau)$ are completely isometrically isomorphic.*

If Conjecture 9.8 is valid on the k-skeleton of Π_n, then the theorem still holds if the covering dimension of X is at most k.

Proof. Clearly (3) implies (2); and (2) implies (1) by Theorem 9.2. Suppose that (1) holds for a homeomorphism γ and an open cover $\{\mathcal{U}_\alpha : \alpha \in S_n\}$. To simplify notation, we may identify Y with X via γ, so that we have $Y = X$ and $\gamma = \mathrm{id}$. If the covering dimension of X is k, then the cover $\{\mathcal{U}_\alpha : \alpha \in S_n\}$ can be refined so that each point of X is contained in at most $k+1$ open sets. Let $\{g_\alpha : \alpha \in S_n\}$ be a partition of unity relative to this cover. This induces a map g of X into the simplex Π_n given by $g(x) = \big(g_\alpha(x)\big)_{\alpha\in S_n}$. In the case of covering dimension k, the image lies in the k-skeleton.

Our hypothesis is that there is a continuous function u from Π_n or its k-skeleton into the unitary group $U(n)$ satisfying the block decomposition condition. Let $v = u \circ g$ map X into $U(n)$. Now if $x \in X$, there is a minimal partition

$$\{1,\dots,n\} = A_1 \dot\cup \dots \dot\cup A_m = B_1 \dot\cup \dots \dot\cup B_m$$

into maximal subsets and an open neighbourhood $\mathcal{U}$ of x so that $\sigma_i|_{\mathcal{U}} = \tau_j|_{\mathcal{U}}$ for $i \in A_s$ and $j \in B_s$, $1 \le s \le m$. The permutations α for which $g_\alpha(x) \ne 0$ respect this block structure. Hence so does the map v. This will ensure that in our construction below, we will always intertwine functions that agree on a neighbourhood of each point.

Let v_{ij} be the matrix coefficients of v. Define operators in $\mathcal{A}(X,\sigma)$ by $T_i = \sum_{j=1}^n \mathfrak{s}_j v_{ij}$. Then since the $\mathfrak{s}_j$'s have pairwise orthogonal range,

$$T_k^* T_i = \sum_{j=1}^n \overline{v_{kj}} v_{ij} = \delta_{ki} I.$$

Hence the T_i's are isometries with pairwise orthogonal ranges. To check the covariance relations, observe that if $v_{ij}(x) \ne 0$, then τ_i and σ_j agree on a neighbourhood of x. That is, $v_{ij}(f\circ\sigma_j) = v_{ij}(f\circ\tau_i)$ for all i, j. Therefore

$$\begin{aligned} fT_i = f\sum_{j=1}^n \mathfrak{s}_j v_{ij} &= \sum_{j=1}^n \mathfrak{s}_j v_{ij}(f\circ\sigma_j) \\ &= \sum_{j=1}^n \mathfrak{s}_j v_{ij}(f\circ\tau_i) = T_i(f\circ\tau_i). \end{aligned}$$

Next observe that $\mathcal{A}(X,\sigma)$ is generated by $C_0(X)$ and $T_iC_0(X)$ for $1\le i\le n$. This is because for $1\le k\le n$,

$$\sum_{i=1}^n T_i\overline{v_{ik}}f = \sum_{i=1}^n\sum_{j=1}^n \mathfrak{s}_j v_{ij}\overline{v_{ik}}f = \sum_{j=1}^n \mathfrak{s}_j f\sum_{i=1}^n v_{ij}\overline{v_{ik}} = \mathfrak{s}_k f.$$

Therefore there is a completely contractive homomorphism of $\mathcal{A}(Y,\tau)$ onto $\mathcal{A}(X,\sigma)$ sending $\mathfrak{t}_i$ to T_i for $1\le i\le n$ which is the identity on $C_0(Y) = C_0(X)$. Likewise there is a completely contractive homomorphism of $\mathcal{A}(X,\sigma)$ onto $\mathcal{A}(Y,\tau)$ which is the inverse on the generators $\mathfrak{s}_j$. Consequently these maps are completely isometric isomorphisms. ∎

Corollary 9.11. *Suppose that* $n = 3$. *Then for two dynamical systems* (X,σ) *and* (Y,τ)*, the following are equivalent:*

(1) (X,σ) *and* (Y,τ) *are piecewise topologically conjugate.*
(2) $\mathcal{A}(X,\sigma)$ *and* $\mathcal{A}(Y,\tau)$ *are isomorphic.*
(3) $\mathcal{A}(X,\sigma)$ *and* $\mathcal{A}(Y,\tau)$ *are completely isometrically isomorphic.*

The dimension 0 case corresponds to totally disconnected spaces. So this result subsumes Corollary 9.4. Since compact subsets of $\mathbb{R}$ have covering dimension 1, we obtain:

Corollary 9.12. *Suppose that* X *has covering dimension* 0 *or* 1. *In particular, this holds when* X *is a compact subset of* $\mathbb{R}$. *Then for two dynamical systems* (X,σ) *and* (Y,τ)*, the following are equivalent:*

(1) (X,σ) *and* (Y,τ) *are piecewise topologically conjugate.*
(2) $\mathcal{A}(X,\sigma)$ *and* $\mathcal{A}(Y,\tau)$ *are isomorphic.*
(3) $\mathcal{A}(X,\sigma)$ *and* $\mathcal{A}(Y,\tau)$ *are completely isometrically isomorphic.*

10. Wandering sets and Recursion

In this section, we examine some topological issues which are needed in the next section on semisimplicity.

Definition 10.1. Let (X,σ) be a multivariable dynamical system. An open set $U\subseteq X$ is said to be (u,v)-*wandering* if

$$\sigma_{uwv}^{-1}(U)\cap U = \varnothing \quad\text{for all}\quad w\in\mathbb{F}_n^+.$$

If U is a (u,v)-wandering set for some u,v in $\mathbb{F}_n^+$, we say that U is a *generalized wandering set*. We may write *v-wandering* instead of $(\varnothing, v)$-wandering.

Observe that if some σ_i is not surjective, then the set $U = X \setminus \sigma_i(X)$ is $(i, \varnothing)$-wandering. If one is interested in when there are no wandering sets, one should restrict to the case in which each σ_i is surjective. In this case, whenever U is (u,v)-wandering, the set $\sigma_u^{-1}(U)$ is $(\varnothing, vu)$-wandering. This open set is non-empty because every σ_u is surjective. In general, the additional generality seems to be needed.

Definition 10.2. Let (X,σ) be a multivariable dynamical system. Given $u,v \in \mathbb{F}_n^+$, we say that a point $x \in X$ is *(u,v)-recurrent* if for every neighbourhood $U \ni x$, there is a $w \in \mathbb{F}_n^+$ so that $\sigma_{uwv}(x) \in U$. We will write *v-recurrent* instead of $(\varnothing, v)$-recurrent.

Again, if each σ_i is surjective, we see that when x is (u,v)-recurrent, then any point y in the non-empty set $\sigma_u^{-1}(x)$ is $(\varnothing, vu)$-recurrent.

The following proposition explains the relationship between recursion and wandering sets in metric spaces.

Proposition 10.3. *Let (X,σ) be a metrizable multivariable dynamical system. There is no (u,v)-wandering set $U \subset X$ if and only if the set of (u,v)-recurrent points is dense in X.*

Proof. If the (u,v)-recurrent points are dense in X, then any open set $U \subset X$ contains such a point, say x_0. Thus there is a word $w \in \mathbb{F}_n^+$ so that $\sigma_{uwv}(x_0) \in U$. Therefore $\sigma_{uwv}^{-1}(U) \cap U \neq \varnothing$. So there are no (u,v)-wandering sets.

Conversely, suppose that there are no (u,v)-wandering sets. Let ρ be the metric on X. For each $x \in X$, define

$$\delta(x) = \delta_{u,v}(x) := \inf_{w \in \mathbb{F}_n^+} \rho(x, \sigma_{uwv}(x)).$$

Observe that δ is upper semicontinuous, i.e. $\{x \in X : \delta(x) < r\}$ is open for all $r \in \mathbb{R}$. Indeed, if $\delta(x_0) < r$ and $\rho(x_0, uwv(\sigma)(x_0)) = r - \varepsilon$, the continuity of our system means that for some $0 < \delta < \varepsilon/2$, $\rho(x, x_0) < \delta$ implies that $\rho(\sigma_{uwv}(x), \sigma_{uwv}(x_0)) < \varepsilon/2$. So by the triangle inequality, $\rho(x, \sigma_{uwv}(x)) < r$; whence $\delta(x) < r$.

Suppose that U is a non-empty open set containing no (u,v)-recurrent points. Then $\delta(x) > 0$ for every $x \in U$. So

$$U = \bigcup_{n \geq 1} \{x \in U : \delta(x) \geq \frac{1}{n}\}.$$

This is a union of closed sets. By the Baire Category Theorem, there is an integer n_0 so that $\{x \in U : \delta(x) \geq \frac{1}{n_0}\}$ has non-empty interior, say V.

Select a ball U_0 contained in V of diameter less than $1/n_0$. We claim that U_0 is (u,v)-wandering. Indeed, for any $x \in U_0$ and any $w \in \mathbb{F}_n^+$, $\rho(x, \sigma_{uwv}(x)) \geq 1/n_0$ and thus $uwv(\sigma)(x)$ lies outside of U_0. That is, $U_0 \cap \sigma_{uwv}^{-1}(U_0) = \varnothing$. ∎

Corollary 10.4. *In a metrizable multivariable dynamical system* (X, σ)*, the following are equivalent:*

(1) *there is no non-empty generalized wandering set in* X
(2) *the set of* (u,v)*-recurrent points are dense for every* $u, v \in \mathbb{F}_n^+$
(3) *each* σ_i *is surjective and the set of* v*-recurrent points is dense in* X *for every* $v \in \mathbb{F}_n^+$*. for every* $v \in \mathbb{F}_n^+$*.*

Proof. The equivalence of (1) and (2) is immediate from the Theorem. It was observed that (1) implies that each σ_i is surjective; and clearly (2) implies as a special case that the set of v-recurrent points is dense in X for every $v \in \mathbb{F}_n^+$.

Conversely, suppose that (3) holds, and fix (u,v) and a point $x_0 \in X$. By surjectivity, there is a point $y_0 \in X$ so that $\sigma_u(y_0) = x_0$. Given a neighbourhood $U_0 \ni x_0$, let $V_0 = \sigma_u^{-1}(U_0)$. Select a vu-recurrent point $y \in V_0$; and let $x = \sigma_u(y)$. If $U \ni x$ is any open set, let $V = \sigma_u^{-1}(U)$. There is a word w so that $\sigma_{wvu}(y) \in V$. Therefore $\sigma_{uwv}(x) \in U$. Thus x is (u,v)-recurrent, and the set of such points is dense in X. So (2) holds. ∎

11. Semisimplicity

We now turn to an analysis of the tensor algebra and the semicrossed product to decide when they are semisimple. In spite of their differences, the answer is again the same. We now apply the results of the previous section to characterize the semisimplicity in our setting.

Theorem 11.1. *Let* (X, σ) *be a multivariable dynamical system. The following are equivalent:*

(1) $\mathcal{A}(X, \sigma)$ *is semisimple.*
(2) $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ *is semisimple.*
(3) *There are no non-empty generalized wandering sets.*

When X *is metrizable, these are also equivalent to*

(4) *Each* σ_i *is surjective and the* v*-recurrent points are dense in* X *for every* $v \in \mathbb{F}_n^+$*.*

Proof. Assume first that there exists a nonempty open set $U \subset X$ and $u, v \in \mathbb{F}_n^+$ so that

$$\sigma_{uwv}^{-1}(U) \cap U = \varnothing \quad \text{for all} \quad w \in \mathbb{F}_n^+.$$

Let $h \neq 0$ be continuous function with support contained in U. Then $(h \circ \sigma_{uwv})h = 0$.

We will show that $\mathcal{A}(X, \sigma)$ is not semisimple. Indeed, the (non-zero) operator $N = \mathfrak{s}_v h \mathfrak{s}_u$ generates a nilpotent ideal. To see this, let $w \in \mathbb{F}_n^+$ and $f \in \mathrm{C}_0(X)$. Compute

$$\begin{aligned} N(\mathfrak{s}_w f)N &= \mathfrak{s}_v h \mathfrak{s}_{uw} f \mathfrak{s}_v h \mathfrak{s}_u \\ &= \mathfrak{s}_{vuwv}(h \circ \sigma_{uwv}) h (f \circ \sigma_v) \mathfrak{s}_u = 0. \end{aligned}$$

Therefore $NAN = 0$ for all $A \in \mathcal{A}(X, \sigma)$. Hence the 2-sided ideal $\langle N \rangle$ generated by N is nilpotent of order 2; and thus $\mathcal{A}(X, \sigma)$ is not semisimple. The same calculation holds in the semicrossed product. So both (2) and (3) imply (1).

Conversely, assume that there are no non-empty wandering sets. We will show that both (2) and (3) hold. As before, we will write $\mathcal{A}$ to denote either the tensor algebra or the semicrossed product. The Jacobson radical of any Banach algebra is invariant under automorphisms. In particular, both of our algebras have the automorphisms α_λ which send the generators $\mathfrak{s}_i$ to $\lambda_i \mathfrak{s}_i$ for each $\lambda \in \mathbb{T}^n$. Integration yields the expectations

$$\Psi_{\mathbf{k}}(a) = \int_{\mathbb{T}^n} \alpha_\lambda(a) \overline{\lambda}^{\mathbf{k}} \, d\lambda$$

for each $\mathbf{a} \in \mathbb{N}_0^n$ onto the polynomials spanned by $\mathfrak{s}_w f$, where $w(\lambda) = \lambda^{\mathbf{k}}$. Consequently, these expectations map the radical into itself.

By way of contradiction, assume that $\operatorname{rad} \mathcal{A}$ contains non-trivial elements. By the previous paragraph, $\operatorname{rad} \mathcal{A}$ will contain a non-zero element of the form

$$Y = \Phi_{\mathbf{a}}(Y) = \sum_{w(\lambda) = \lambda^{\mathbf{k}}} \mathfrak{s}_w h_w =: \sum_{j=1}^{p} \mathfrak{s}_{w_j} h_j$$

for some $\mathbf{a} \in \mathbb{N}_0^n$. Since $Y \neq 0$, we may suppose that $h_1 \neq 0$. By multiplying Y on the right by a function, we may suppose that $h_1 \geq 0$ and $\|h_1\| > 1$.

We will look for an element of the form

$$Q = \big(I + \sum_{k \geq 1} 2^{-k} \mathfrak{s}_{v_k}\big) Y = \sum_{k \geq 0} 2^{-k} \sum_{j=1}^{p} \mathfrak{s}_{w_{k,j}} h_j$$

where $w_{k,j} = v_k w_j$. The goal will be to show that Q is not quasinilpotent, so contradicting the fact that Y is in the radical.

When $n = 1$, Y has only one term and the coefficients in Q and all of its powers are non-negative functions. Here there is no cancellation of terms, so the size of a single term in the product Q^m yields a useful lower bound for the norm.

However when $n \geq 2$, this is not the case. Instead, we can arrange to choose the words v_k so that the distinct terms in a product Q^m will be distinct words in $\mathbb{F}_n^+$. This will be accomplished if the word v_k begins with the sequence $t_k = 21^k2$, the sequence consisting of k 1s separating two 2s. For such v_k, consider the terms in a product Q^m. They will have the form $\mathfrak{s}_u g_u$ where u is a product of m words $u = w_{k_1,j_1} \dots w_{k_m,j_m}$. The first few letters of u will be t_{k_1}, uniquely identifying k_1 and w_{k_1,j_1} will be the first $|v_{k_1}| + |a|$ terms of u. Peel that term off and repeat. Since the product determines the terms, in order, it follows that distinct products yield distinct words. So again, there is no cancellation of terms. Therefore the size of a single term in the product will yield a lower bound for the norm of Q.

The plan of attack is to show that

$$Q^{2^k-1} = 2^{-n_k}\mathfrak{s}_{u_k} g_k + \text{other terms}$$

where $\|g_k\| > 1$ and $n_k = 2^k - k - 1$. In computing the next power, we see that

$$\begin{aligned} Q^{2^{k+1}-1} &= Q^{2^k-1} Q Q^{2^k-1} \\ &= (2^{-n_k}\mathfrak{s}_{u_k} g_k)(2^{-k}\mathfrak{s}_{w_{k,1}} h_1)(2^{-n_k}\mathfrak{s}_{u_k} g_k) + \text{other terms} \\ &= 2^{-2n_k-k}\mathfrak{s}_{u_k w_k u_k}(g_k \circ \sigma_{w_k u_k})(h_1 \circ \sigma_{u_k}) g_k + \text{other terms} \\ &= 2^{-n_{k+1}}\mathfrak{s}_{u_{k+1}} g_{k+1} + \text{other terms.} \end{aligned}$$

Provided that $\|g_k\| > 1$ for all $k \geq 1$, we obtain that

$$\operatorname{spr}(Q) = \lim_{k\to\infty} \|Q^{2^k-1}\|^{1/(2^k-1)} \geq \lim_{k\to\infty} 2^{-\frac{2^k-k-1}{2^k-1}} = \frac{1}{2} > 0.$$

The preceding calculation indicates where we should look. One has formulae

$$w_{k,1} = v_k w_1, \quad v_0 = \varnothing, \quad \text{and} \quad u_{k+1} = u_k w_{k,1} u_k \quad \text{for} \quad k \geq 0,$$

and v_k begins with t_k for $k \geq 1$. We choose $u_0 = \varnothing$ so that $u_1 = w_1$ to start the induction. We also have functions $g_1 = h_1$ and

$$g_{k+1} = (g_k \circ \sigma_{w_{k,1} u_k})(h_1 \circ \sigma_{u_k}) g_k \quad \text{for} \quad k \geq 1.$$

We need to ensure that $\|g_k\| > 1$ for all $k \geq 1$.

To this end, let $V_1 = \{x \in X : h_1(x) > 1\}$. Our task will be accomplished if we can find non-empty open sets $V_{k+1} \subset V_k$ for $k \geq 1$ so that

$$\sigma_{w_k u_k}(V_{k+1}) \subset V_k \quad \text{and} \quad \sigma_{u_k}(V_k) \subset V_1 \quad \text{for all} \quad k \geq 1.$$

Indeed, if we have these inclusions, we can show by induction that $g_k > 1$ on V_k. Clearly this is the case for $k = 1$. Assuming the properties outlined in the previous paragraph,

$$\sigma_{u_k}(V_{k+1}) \subset \sigma_{u_k}(V_k) \subset V_1;$$

and thus $h_1 \circ \sigma_{u_k} > 1$ on V_{k+1}. Similarly, $\sigma_{w_k u_k}(V_{k+1}) \subset V_k$, and so $g_k \circ \sigma_{w_k u_k} > 1$ on V_{k+1}. Finally $g_k > 1$ on $V_k \supset V_{k+1}$. Therefore the product yields $g_{k+1} > 1$ on V_{k+1}.

To construct V_{k+1}, observe that there are no $(t_{k+1}, w_1 u_k)$-wandering sets. In particular, there is a $v'_k \in \mathbb{F}_n^+$ so that

$$V_{k+1} := \sigma^{-1}_{t_k v'_k w u_k}(V_k) \cap V_k \neq \varnothing.$$

With this choice (and recalling that $w_{k,1} = t_k v'_k w_1$), we have that

$$\sigma_{w_{k,1} u_k}(V_{k+1}) \subset V_k \quad \text{and} \quad V_{k+1} \subset V_k.$$

Lastly,

$$\sigma_{u_{k+1}}(V_{k+1}) = \sigma_{u_k} \sigma_{w_{k,1} u_k}(V_{k+1}) \subset \sigma_{u_k}(V_k) \subset V_1.$$

This completes the induction. We see that the radical must be $\{0\}$, and so $\mathcal{A}$ is semisimple.

The equivalence of (1) and (4) in the metric case is given by Corollary 10.4. ∎

References

[1] W. Arveson, *Operator algebras and measure preserving automorphisms*, Acta Math. **118**, (1967), 95–109.
[2] W. Arveson, *Subalgebras of C*-algebras*, Acta Math. **123** (1969), 141–224.
[3] W. Arveson, *Subalgebras of C*-algebras III*, Acta Math. **181** (1998), 159–228.
[4] W. Arveson, *The noncommutative Choquet boundary*, preprint, 2007.
[5] W. Arveson and K. Josephson, *Operator algebras and measure preserving automorphisms II*, J. Funct. Anal. **4**, (1969), 100–134.
[6] D. Blecher and C. LeMerdy, *Operator algebras and their modules-an operator space approach*, Clarendon Press, Oxford, 2004
[7] D. Blecher, Z.-J. Ruan and A. Sinclair, *A characterization of operator algebras*, J. Funct. Anal. **89** (1990), 188–201.
[8] J. Bunce, *Models for n-tuples of non-commuting operators*, J. Funct. Anal. **57** (1984), 21–30.
[9] K. Davidson and E. Katsoulis, *Isomorphisms between topological conjugacy algebras*, preprint.
[10] A. Donsig, A. Katavolos, A. Manoussos, *The Jacobson radical for analytic crossed products*, J. Funct. Anal. **187**, (2001), 129–145.

[11] R. Douglas and V. Paulsen, *Hilbert Modules for Function Algebras*, Res. Notes in Math. **217**, Longman, Harlow/New York, 1989.

[12] M. Dritschel and S. McCullough, *Boundary representations for families of representations of operator algebras and spaces*, J. Operator Theory **53** (2005), 159–167.

[13] S. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc. **68** (1978), 300–304.

[14] A. Frahzo, *Models for non-commuting operators*, J. Funct. Anal. **48** (1982), 1–11.

[15] K. Fritzsche and H. Grauert, *From holomorphic functions to complex manifolds*, Graduate Texts in Mathematics, **213**, Springer-Verlag, New York, 2002.

[16] N. Fowler, P. Muhly and I. Raeburn, *Representations of Cuntz-Pimsner algebras*, Indiana Univ. Math. J. **52** (2003), 569-605.

[17] M. Hamana, *Injective envelopes of operator systems*, Publ. Res. Inst. Math. Sci. **15** (1979), 773-785.

[18] D. Hadwin and T. Hoover, *Operator algebras and the conjugacy of transformations.*, J. Funct. Anal. **77** (1988), 112–122.

[19] M. Ionescu, *Mauldin-Williams graphs, Morita equivalence and isomorphisms*, Proc. Amer. Math. Soc. **134** (2006), 1087–1097.

[20] E. Katsoulis and D. Kribs, *Isomorphisms of algebras associated with directed graphs*, Math. Ann. **330** (2004), 709–728.

[21] E. Katsoulis and D. Kribs, *The C^*-envelope of the tensor algebra of a directed graph*, Integral Equations Operator Theory, to appear.

[22] E. Katsoulis and D. Kribs, *Tensor algebras of C^*-correspondences and their C^*-envelopes*, J. Funct. Anal. **234** (2006), 226–233.

[23] T. Katsura, *On C^*-algebras associated with C^*-correspondences*, J. Funct. Anal. **217** (2004), 366–401.

[24] T. Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. I. Fundamental results*, Trans. Amer. Math. Soc. **356** (2004), 4287–4322.

[25] E.C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists*, London Math. Soc. Lecture Note Series, **210**, Cambridge University Press, Cambridge, 1995.

[26] M. McAsey and P. Muhly, *Representations of non-selfadjoint crossed products*, Proc. London Math. Soc. (3) **47** (1983), 128–144.

[27] M. Mimura and H. Toda, *Topology of Lie groups. I, II*, Transl. Math. Monographs **91**, American Mathematical Society, Providence,1991.

[28] P. Muhly, *Radicals, crossed products, and flows*, Ann. Polon. Math. **43** (1983), 35–42.

[29] P. Muhly and B. Solel, *Hilbert modules over operator algebras*, Mem. Amer. Math. Soc. **117**, no. **559**, Amer. Math. Soc., Providence, RI, 1995.

[30] P. Muhly and B. Solel, *An algebraic characterization of boundary representations*, Nonselfadjoint operator algebras, operator theory, and related topics, Oper. Theory Adv. Appl. **104**, 189–196, Birkhauser, Basel, 1998.

[31] P. Muhly and B. Solel, *Tensor algebras over C^*-correspondences: representations, dilations and C^*-envelopes*, J. Funct. Anal. **158** (1998), 389–457.

[32] P. Muhly and B. Solel, *Tensor algebras, induced representations, and the Wold decomposition*, Can. J. Math. **51** (1999), 850–880.

[33] P. Muhly and M. Tomforde, *Topological quivers*, Internat. J. Math. **16** (2005), 693–755.

[34] K. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York, 1967.

[35] S. Parrott, *Unitary dilations for commuting contractions*, Pacific J. Math. **34** (1970), 481–490.

[36] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, Cambridge, 2002.

[37] J. Peters, *Semicrossed products of C^*-algebras*, J. Funct. Anal. **59** (1984), 498–534.

[38] M. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$*, Free probability theory (Waterloo, ON, 1995), 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.

[39] G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer. Math. Soc. **316** (1989), 523–536.

[40] S. Power, *Classification of analytic crossed product algebras*, Bull. London Math. Soc. **24** (1992), 368–372.

[41] A. Sinclair, *Homomorphisms from $C_0(\mathbb{R})$*, J. London Math. Soc. **11** 1975, 165–174.

[42] N. Varopoulos, *On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory*, J. Funct. Anal. **16** (1974), 83–100.

Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: krdavids@uwaterloo.ca

Dept. Math., East Carolina University, Greenville, NC 27858, USA
E-mail address: KatsoulisE@mail.ecu.edu