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UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO

PMATH 331/AM 331 **Real Analysis and Applications**

Midterm Examination

October 22, 2003 **Time: 2 hours**

Instructor: R. André

1. Answer the questions in the spaces provided, using the backs of pages for overflow or rough work. The exam is out of 65, with the marks for each question shown in the margin.
2. Justify all statements **carefully**. When invoking a result or statement proven in class, state the conditions which allow the conclusion of the statement to hold true.
3. If you finish early, take time to polish up your solutions and tighten up your proofs. No extra points are given for handing in your copy early, but some may be lost for poor presentation.

Question	Mark	
1	10 /10	
2	4 /6	14
3	6 /6	20
4	5 /8	25
5	6 /6	31
6	5 /8	36
7	9 /9	45
8	12 /12	
Total	57 /65	

[10] 1. (a) Define "the set U is open in $\mathbf{R}^n$".

The set $U \subseteq \mathbb{R}^n$ is open if

2 $\forall \bar{x} \in U$, the open ball $B_r(\bar{x})$ is entirely contained in U, for some $r > 0$, $r \in \mathbb{R}$.

2

(b) Show that the set $\mathbf{R}\backslash\{0\}$ is open in $\mathbb{R}$.

It is sufficent to show the compliment say U is closed.

$U = \{0\}$. Let $\{x_i\}$ be any seq in U. It is clear

That $\{x_i\} = \{0, 0, 0, 0, \dots\} \rightarrow 0$.

Since U contains all its limit pts, it is closed by defn.

3/3

Since U is closed, its compliment $U' = \mathbb{R}\backslash\{0\}$ is open. $\square$

(c) Is the set $\{(x, y, z) : ||(x, y, z)|| \geq 3\}$ open, closed or neither? If it is open or closed prove it. If it is neither show why.

Closed.

Let $\hat{a}$ be any limit pt of $A = \{(x,y,z) : \|(x,y,z)\| \geq 3\}$.

Claim: $\bar{a} \in A$.

Assume $\bar{a} \notin A$.

Let $\{\bar{x}_n\}$ be any seq in A which converges to $\bar{a}$.

Then $\forall \varepsilon > 0$ $\exists n > N$ st $\|\bar{x}_n - \bar{a}\| < \varepsilon$.

Since $\bar{a} \notin L$, $\|\bar{a}\| < 3$. Choose $\varepsilon < 3 - \|a\|$.

4/4

~~Then~~ Since $\bar{x}_n \to \bar{a}$, $\|\bar{x}_n - a\| < \varepsilon$ but by Δ,

$\|\bar{x}_n - \bar{a}\| \geq \|x_n\| - \|a\|$

$\geq 3 - \|a\|$

$= \varepsilon$

CONTRADICTION!

The assumption that $\bar{a} \notin A$ caused a contradiction, thus any limit pt a of A is in A,

$\therefore$ by defn, $A = \{(x,y,z) : \|(x,y,z)\| \leq 3\}$ is closed. $\square$

[6] 2. (a) Define a Cauchy sequence in $\mathbf{R}^n$.

A sequence $\{\bar{a}_n\}$ is Cauchy in $\mathbb{R}^n$ if $\forall \varepsilon > 0$ $\exists m, n > N$ st $\|x_m - x_n\| < \varepsilon$.

2/2

(b) Prove that every convergent sequence in $\mathbf{R}^n$ is a Cauchy sequence.

Let $\{\bar{a}_n\}$ be a convergent seq in $\mathbb{R}^n$.

Then let $\varepsilon > 0$. By defn, for $n > N$, $\|a_n - L\| < \varepsilon/2$.

For any $m > N$,

$$\|a_n + a_m - a_m - L\| < \varepsilon/2$$

rearrange

$$\|a_n - a_m - (L - a_m)\| < \varepsilon/2$$

2/4

See notes

by Δ,

$$\|a_n - a_m\| - \|L - a_m\| < \varepsilon/2$$

$$\|a_n - a_m\| < \varepsilon/2 + \|L - a_m\|$$

$$< \varepsilon/2 + \varepsilon/2$$

X

$$< \varepsilon$$

$\therefore$ by defn of convergence, Cauchy seqs in $\mathbb{R}^n$ converge $\square$

[6] 3. (a) What does it mean to say that a subset of $\mathbf{R}^n$ is complete?

A subset S of $\mathbb{R}$ is complete if all Cauchy sequences in S converge in S.

ie. $S \subseteq \mathbb{R}^n$ is complete if $\forall \{x_n\}$ st $\{x_n\}$ is Cauchy, then $\lim_{n \to \infty} x_n \in S$.

2/2

(b) Consider the set $A = \{\sin n : n = 1, 2, 3, \dots\}$. Is A complete? Justify.

No, A is not complete.

From lecture, we know A is dense over $[-1, 1]$, that is, $\forall x \in [-1, 1]\ \exists \{x_n\}$ in A st $x_n \to x$.

However, some of these x are NOT in A. eg $\sin(\pi) \notin A$.

So pick $\{x_n\}$ st $x_n \to \cos\pi$. Since all convergent sequences are Cauchy, $\{x_n\}$ is Cauchy.

However, $\{x_n\}$ is Cauchy but $\{x_n\}$ does NOT converge in A. $\therefore$ by defn, A is NOT complete.

[8] 4. (a) Define compact set.

A set S is compact if every seq $\{x_n\}$ in S has a convergent subseq in S.

2/2

(b) State the Heine-Borel theorem.

The Heine-Borel thm states that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

2/2

(c) Let $B = \{\frac{1}{n} : n = 1, 2, 3, \ldots\} \cup \{0\}$. Define the function $f : B \to \{0,1\}$ as follows: $f(\frac{1}{n}) = 0$ if n is even, $f(\frac{1}{n}) = 1$ if n is odd, $f(0) = 0$. Let $A = \{(x, f(x)) : x \in B\}$. Is A compact? Justify all steps.

4

A is NOT compact.

Compact sets, by Heine-Borel, are closed and bounded.

Claim: A is NOT bounded. X

A is bded iff $\bar{x} \in A \Rightarrow \bar{x} \in B_r(\bar{0})$ for some r.

Spse A is bded. Then $\exists$ such an r.

So $\bar{x} \in A \Rightarrow \bar{x} \in B_r(\bar{0})$.

ie $\|\bar{x}\| < r$

however, pick $\bar{x}$ st $\bar{x} = (r, f(r))$.

Since $r > 0$, $f(r) > 0$.

Then $\|x\| = \sqrt{r^2 + [f(r)]^2} > \sqrt{r^2} > r$ α

CONTRADICTION!

Since $\forall r > 0$ we can find an $\bar{x}$ st $\|\bar{x}\| > r$,

A cannot be bounded.

$\therefore$ A is NOT bounded, then by Heine-Borel,

A is NOT compact.

[6] 5. Show that a complete subset of $\mathbf{R}^n$ is closed.

6

Let $S \subseteq \mathbb{R}^n$ be complete.

Need to show every convergent seq in S converges in S.

Let $\{x_n\}$ be any convergent sequence in S.

By a thm, any convergent seq is Cauchy, so $\{x_n\}$ is Cauchy.

Since S is complete, any Cauchy seq in S converges in S,

So since $\{x_n\}$ is Cauchy in S, it converges in S.

Since $\{x_n\}$ was any convergent seq in S, just showed that any convergent seq in S converges in S.

$\therefore$ by defn of closed, S is closed, any complete subset of $\mathbb{R}^n$ is closed $\square$

[8] 6. (a) What does it mean to say that a sequence of real numbers $\{x_n\}$ is a contractive sequence.

1.5

A seq of real numbers is contractive if $\exists c$ st $0<c<1$ and $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$.

(b) If $x_1 > 0$ and $x_{n+1} = 1/(2+x_n)$ for $n \geq 1$, show that $\{x_n\}$ is a contractive sequence.

3.5

$x_1 > 0$

$x_2 = \frac{1}{2+x_1}$

$x_3 = \frac{1}{2+\frac{1}{2+x_1}}$

$x_4 = \frac{1}{2+\frac{1}{2+\frac{1}{2+x_1}}}$

$|x_4 - x_3| = \left|\frac{1}{2+x_3} - \frac{1}{2+x_2}\right|$

$= \left|\frac{2-x_2-2+x_3}{(2-x_3)(2-x_2)}\right|$

$= \left|\frac{x_3-x_2}{4-2x_3-2x_2+x_2x_3}\right|$

we know $x_n > 0$ since $x_1 > 0$.

define $d_k = |x_k - x_{k-1}|$

$d_k = \left|\frac{1}{2-x_{k-1}} - x_{k-1}\right|$

$|x_4 - x_3| = \left|\frac{1}{2+x_3} - x_3\right|$

$\cancel{= \left|\frac{2x_3 - x_3^2}{2+x_3}\right|}$

$= \left|(2+x_3)\frac{x}{1}\right|$

[9] 7. (a) Define "$f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at **a**".

$f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\bar{a} \in \mathbb{R}^n$ if

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = f(\bar{a})$$

(b) Define a Lipschitz function.

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if $\exists\, C \in \mathbb{R}^+$ st

$$\|f(\bar{x}) - f(\bar{y})\| \leq C\|\bar{x} - \bar{y}\| \quad \forall \bar{x}, \bar{y} \in \mathbb{R}^n$$

(c) Show that a Lipschitz function is continuous.

let $f: \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz.

then $\exists\, C \in \mathbb{R}^+$ st $\|f(\bar{x}) - f(\bar{y})\| \leq C\|\bar{x} - \bar{y}\| \quad \forall \bar{x}, \bar{y} \in \mathbb{R}$

Claim: $\forall \epsilon > 0\ \exists \delta > 0$ st $\|\bar{x} - \bar{a}\| < \delta \Rightarrow \|f(\bar{x}) - f(\bar{a})\| < \epsilon$

let $\epsilon > 0$. Pick $\delta = \frac{\epsilon}{C}$.

Then $\|\bar{x} - \bar{a}\| < \delta$

$\Rightarrow \|f(\bar{x}) - f(\bar{a})\| \leq C\|\bar{x} - \bar{a}\| < C\delta$ since f Lipschitz and $C > 0$.

so $\|f(\bar{x}) - f(\bar{a})\| < C\delta = C\left(\frac{\epsilon}{C}\right) = \epsilon$

so $\|f(\bar{x}) - f(\bar{a})\| < \epsilon$

and f is convergent by defn. $\square$

[12] 8. (a) State the Monotone convergence theorem.

The monotone convergence Thm states that given a seq $\{a_n\}$

if H_1: $\{a_n\}$ is bounded

H_2: $\{a_n\}$ is monotone increasing (decreasing)

then C_1: $\{a_n\}$ converges

C_2: $\{a_n\}$ converges to its supremum (infimum).

(b) Let $x_1 > 2$ and $x_{n+1} = 2 - 1/x_n$ for $n = 1, 2, 3, \ldots$. Show that the sequence of real numbers $\{x_n\}$ is monotone.

~~$x_1 > 2$~~

~~$x_2 = 2 - 1/x_n < 2$ since $x_1 > 2$~~

~~Spse true for x_k.~~

~~$x_{k+1} = 2 - \frac{1}{x_k} = \frac{2x_k - 1}{x_k}$ (*)~~

~~Claim: $x_n > 1 \quad \forall n$~~

~~$x_1 > 2 > 1$~~

~~Assume $x_j > 1$. Show $x_{j+1} > 1$.~~

~~$x_{j+1} = 2 - \frac{1}{x_j}$ now since $x_j > 1$, $\frac{1}{x_j} < 1$~~

~~$\therefore 2 - \frac{1}{x_j} > 1$ by above~~

~~so $x_{j+1} > 1$ and all $x_n > 1$.~~

~~back to (*), $x_{k+1} \leq x_k \left[\frac{2x_k - 1}{x_k}\right]$ since $x_k > 1$~~

~~$= 2x_k - 1 < x_k$ since $x_k < 2$~~

OMIT

Solution on back of previous page.

(c) Suppose that we have determined that the sequence in part (b) is bounded above by 2 and below by 1. Does this sequence converge? Why? If the sequence converges find its limit. Justify all steps carefully.

Yes $\{x_n\}$ from (b) will converge given (c).

Using the Monotone convergen thm,

$\{x_n\}$ is bounded and $\{x_n\}$ is monotone decreasing.

$\therefore$ by MCT, $\{x_n\}$ converges.

$$x_{n+1} = 2 - \frac{1}{x_n}$$

let $\lim_{n\to\infty} x_n = L$

so, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left[2 - \frac{1}{x_n}\right]$

$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = L$ so

$L = \lim_{n\to\infty} 2 - \lim_{n\to\infty} \frac{1}{x_n}$ by limit props.

$L = 2 - \frac{1}{L}$ by limit props

$L^2 = 2L - 1$

$L^2 - 2L + 1 = 0$

$(L-1)^2 = 0$ ✓

$\therefore$ the limit is 1.

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