AMath/PMath 331
Assignment 7
Due Friday December 2

1. (a) Let $u(r,\theta) = (3 - 4r^2 + r^4) + (8r^2 - 8r^4)\sin^2\theta + 8r^4\sin^4\theta$. Compute $\Delta u(r,\theta)$.

   (b) Let $u(r,\theta) = \log r$. Compute Δu and $u(1,\theta)$. Explain why u is not a solution of the heat equation for the boundary function $f(\theta) = 0$.

2. (a) Prove that $\dfrac{1-r}{1+r} \le 2\pi P(r,\theta) \le \dfrac{1+r}{1-r}$.

   (b) Let f be a *positive* continuous 2π-periodic function with harmonic extension $u(r,\theta)$. Prove that $u(r,\theta) \ge 0$.

   (c) Again $f \ge 0$. Prove *Harnack's inequality*: $\Big(\dfrac{1-r}{1+r}\Big)u(0,0) \le u(r,\theta) \le \Big(\dfrac{1+r}{1-r}\Big)u(0,0)$.

3. Which of the following are the Fourier series of a continuous function? Explain—but do not try to calculate the sum.

   (a) $2 + \sum_{n=1}^{\infty} \frac{1}{n^3}\cos n\theta + \frac{1}{3^n}\sin n\theta$.

   (b) $1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cos n\theta + \frac{1}{2^n}\sin n\theta$. **Hint:** look at $\lim\limits_{r\to 1} u(r,\pi)$.

4. (a) Compute the Fourier series of the function $f(\theta) = |\sin\theta|$.
   Hint: use symmetry and the addition formula for $\sin(n\theta \pm \theta)$.

   (b) Show that the Fourier series converges uniformly.

   (c) Hence evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$. **Hint:** what value of θ is needed here?

5. (a) Suppose $u(x,y)$ is a solution of the heat problem on $\overline{\mathbb{D}}$ (given in rectangular coordinates). Let $D_R(x_0,y_0)$ be a small disk of radius R contained inside $\mathbb{D}$. Establish the *mean value property*:

   $$u(x_0,y_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta.$$

   Hint: The restriction of u to $\overline{D_R(x_0,y_0)}$ is the solution to the heat problem on this disk. Use the Poisson formula for the value at the centre of this disk.

   (b) Suppose that $u(x,y)$ is a continuous function on $\overline{\mathbb{D}}$ that satisfies the mean value property. Prove that if $u(x,y)$ attains its maximum value at an interior point, then it must be constant. **Hint:** consider the value of u on concentric circles around points attaining the maximum.

6. **Bonus.** Prove that a continuous function on $\mathbb{D}$ satisfing the mean value property is harmonic. **Hint:** Fix a point (x_0,y_0) in $\mathbb{D}$ and let $D_R(x_0,y_0)$ be a small disk contained inside $\mathbb{D}$. Let $v(x,y)$ be the solution of the steady-state heat problem on $D_R(x_0,y_0)$ that agrees with u on the boundary circle. Then $u-v$ satisfies the mean value property. Show that $u = v$, and hence deduce that $\Delta u(x_0,y_0) = 0$.