AMath/PMath 331
Assignment 6
Due Monday November 21

1. Let $Sx = \frac{1}{2}\big(x + \sqrt{x^2+1}\,\big)$ and $Tx = \tan^{-1} x$ on $\mathbb{R}$.
   (a) Show that $|Sx - Sy| < |x-y|$ and $|Tx - Ty| < |x-y|$ for $x, y \in \mathbb{R}$.
   (b) Show that S and T are not contraction maps.
   (c) Show that T has a unique fixed point, and that S has no fixed point.

2. (a) Show that $f(x) = x^3 + x + 1$ has exactly one real root.
   (b) Use Newton's method to approximate it to eight decimal places. (You can use a computer to do the calculations.) Show your error estimates.

3. For differential equation

$$y^{(3)} + y'' - x(y')^2 = e^x, \quad y(0) = 1,\ y'(0) = -1, \text{ and } y''(0) = 0$$

   convert the DE into a first-order vector-valued DE and then into a fixed-point problem. You do not have to solve it.

4. Consider the DE $y' = 1 + xy$ and $y(0) = 0$ on $[-1, 1]$.
   (a) Show that the associated integral operator is a contraction mapping.
   (b) Find a convergent power series expansion for the unique solution.
   (c) Use the Global Picard Theorem to show that there is a unique solution on $[-b, b]$ for any $b < \infty$. Hence deduce that there is a unique solution on $\mathbb{R}$.

5. Consider $f'(x) + f(x)^2 = 4xf(x) - 4x^2 + 2$ for $x \in \mathbb{R}$ and $f(0) = 2$.
   (a) Convert this to a first order vector valued DE.
   (b) Show that this DE satisfies a local Lipschitz condition on a smaller region; and hence deduce that there is a local solution.
   (c) Solve this DE explicitly, and find the maximal continuation of the solution. **Hint:** Find the DE satisfied by $g(x) = f(x) - 2x$ and solve it.

6. Consider the DE $f(x)f'(x) = 1$ and $f(0) = a$ for $x \in \mathbb{R}$.
   (a) Solve this equation explicitly.
   (b) Show that there is a unique solution on an interval about 0 if $a \neq 0$ but that it extends to only a proper subset of $\mathbb{R}$, even though the solution does not blow up. Why does this not contradict the Continuation Theorem?
   (c) Show that there are two solutions when $a = 0$ valid on $[0, \infty)$. Why does this not contradict the Local Picard Theorem?