AMath/PMath 331
Assignment 5
Due Monday November 7

1. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ converges uniformly on the whole real line.

2. Show that if $f \in \mathrm{C}[0,1]$, then $\|B_n f\|_\infty \le \|f\|_\infty$ for all $n \ge 0$.

3. Look at the proof of the Weierstrass Theorem using the function $f(x) = \left|x - \frac{1}{2}\right|$. For $\varepsilon = \frac{1}{2}10^{-3}$, what value of N is required to obtain $\|f - B_N f\|_\infty < \varepsilon$?

4. Suppose that $f \in \mathrm{C}^1[0,1]$. Prove that f is a C^1-limit of polynomials. i.e. find polynomials p_n so that $\|f - p_n\|_{\mathrm{C}^1} = \max\{\|f - p_n\|_\infty, \|f' - p_n'\|_\infty\}$ tends to 0. **Hint:** Approximate f' by polynomials and integrate.

5. Suppose that $f \in \mathrm{C}[0,1]$ satisfies $\displaystyle\int_0^1 f(x)x^n\,dx = 0$ for all integers $n \ge 0$.
   Prove that $f = 0$.
   Hint: Use polynomial approximation to show that $\displaystyle\int_0^1 |f(x)|^2\,dx = 0$.

6. (a) If a, b, c are real numbers, show that the quadratic polynomial p such that $p(0) = a$, $p(\frac{1}{2}) = b$ and $p(1) = c$ satisfies $\|p\|_{[0,1]} \le |a| + |b| + |c|$.
   Hint: find three quadratics which take the values $\{0,0,1\}$ on $\{0, \frac{1}{2}, 1\}$, and combine.
   (b) If $f \in \mathrm{C}[0,1]$, show that there is a sequence of polynomials p_n converging uniformly to f on $[0,1]$ such that $p_n(0) = f(0)$, $p_n(\frac{1}{2}) = f(\frac{1}{2})$ and $p_n(1) = f(1)$.
   Hint: start with a sequence of polynomials converging to f, and modify it using (a) to make the polynomials agree with f at the three points.
   (c) **Bonus.** Show that there is a sequence of polynomials p_n converging to f uniformly on $[0,1]$ such that $p_n(\frac{k}{n!}) = f(\frac{k}{n!})$ for $0 \le k \le n!$.

7. Find the closest cubic polynomial to $|x|$ on $[-1,1]$. How close is it?