AMath/PMath 331
Assignment 4
Due Friday October 28

1. Let $f_n(x) = nxe^{-nx}$ on $[0, \infty)$.
   (a) Prove that this sequence converges pointwise.
   (b) Is the convergence uniform?

2. Let $g_n(x) = \dfrac{x}{1 + nx^2}$ on $[0, \infty)$.
   (a) Prove that this sequence converges pointwise.
   (b) Is the convergence uniform?

3. Let $h_n(x) = \dfrac{n + x}{4n + x}$ on $[0, \infty)$.
   (a) Prove that this sequence converges pointwise.
   (b) Prove that it converges uniformly on $[0, N]$.
   (c) Show that it does not converge uniformly on $[0, \infty)$.

4. Let K be a compact subset of $\mathbb{R}^n$. Suppose that $f_n \in \mathrm{C}(K)$ are all Lipschitz with Lipschitz constant C, and they converge uniformly to f. Prove that f is Lipschitz with Lipschitz constant C.

5. Suppose that $f(x, t)$ is continuous on $[a, b] \times [c, d]$. Define $F(x) = \int_c^d f(x, t)\, dt$. Show that F is continuous. **Hint:** f is uniformly continuous.

6. Define the L^1 norm on $\mathrm{C}[0, 2]$ by $\|f\|_1 = \int_0^2 |f(x)|\, dx$.
   (a) Prove that this is a norm.
   (b) Let

$$f_n(x) = \begin{cases} x^n & \text{if} \quad 0 \le x \le 1 \\ 1 & \text{if} \quad 1 \le x \le 2 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad 1 \le x \le 2 \end{cases}.$$

   Show that the sequence $f_n(x)$ converges in the L^1 norm to $f(x)$.
   (c) **Bonus.** Hence show that $\mathrm{C}[0, 2]$ is not complete in the L^1 norm.
   Hint: why is there no *continuous* function which is the limit of (f_n)?

What is covered on the Midterm on Tuesday October 25?

Tuesday October 25, 7:00–8:30. Arrive 10 minutes early to get set up.
If you are enrolled in AMATH 331, go to room MC 4020.
If you are enrolled in PMATH 331, go to room MC 4045.
Spread out so that no-one is sitting right beside you. The exam is the same in both rooms. This is just a method to split the class properly.

- You should know all the definitions and how to use them.
- You should know the statements of all theorems that have names.
- You should know the proofs of the following theorems:

  (1) The Monotone Convergence Theorem.
  (deduced from the Least Upper Bound Principle.)

  (2) Lemma *A compact subset of* $\mathbb{R}^n$ *is closed and bounded.*
  (This is the easy half of the Heine-Borel Theorem.)

  (3) Cantor Intersection Theorem.

  (4) Extreme Value Theorem.

- You should be able to do problems like the homework exercises (although because of time considerations, the midterm problems will be a bit shorter).