AMath/PMath 331
Assignment 3
Due Friday October 14

1. Consider a function defined on $\mathbb{R}^2$ by
$$f(x,y) = \begin{cases} 0 & \text{if} \quad y \le 0 \\ 0 & \text{if} \quad y \ge x^2 \\ \sin\left(\frac{\pi y}{x^2}\right) & \text{if} \quad 0 < y < x^2 \end{cases}.$$
   (a) Show that f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.
   (b) Show that f is not continuous at the origin.
   (c) Show that the restriction of f to any straight line through the origin is continuous.

2. Suppose $f : \mathbb{R}^n \to \mathbb{R}^+$ is a positive continuous function such that $\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 0$. Prove that f attains its maximum value.
Hint: there is a large R so that $f(\mathbf{x}) < f(\mathbf{0})/2$ when $\|\mathbf{x}\| > R$.

3. Prove that $4 \sin x + 3 \cos x = x$ has at least three real solutions.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that
$$f(x,y) = f(x+2,y) = f(x,y+5) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2.$$
   (a) Prove that f attains its maximum and minimum values.
   Hint: why is it enough to consider $(x,y) \in [0,2] \times [0,5]$?
   (b) Prove that f is uniformly continuous.
   Hint: why is it enough to consider $(x,y) \in [-2,2] \times [-5,5]$?

5. (a) Show that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.
   (b) Show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 2]$.
   (c) Show that if $0 \le x < y \le 4^{-n-1}$, then $f(y) - f(x) > 2^n(y - x)$.
   Hint: Mean Value Theorem.
   (d) **Don't hand in—just think about it.** Why doesn't (c) contradict (b)?

6. **Bonus.** Let f be a continuous function from the closed ball in $\mathbb{R}^2$, namely $\overline{B} = \{x \in \mathbb{R}^2 : \|x\| \le 1\}$, into $\mathbb{R}$. Show that f cannot be not one to one.