AMath/PMath 331
Assignment 2
Due Monday October 3

1. (a) For each of the following sets, provide a sketch. State whether it is open, closed or neither. If it is *not closed*, identify a limit point of the set which is not in the set. If it is *not open*, identify a point in the set which is a limit point of the complement.

   (i) $A = \{(e^{-x}\cos x, e^{-x}\sin x) : x \geq 0\} \cup \{(x, 0) : 0 \leq x < 1\}$.

   (ii) $B = \{(r, s) : r, s \in \mathbb{Q},\ r^2 + s^{-2} < 1\}$.

   (iii) $C = \{(x, \sin y) : x^2 < 4,\ y > 0\}$.

   (iv) $D = \{(x, y) : x \neq 0,\ |y| < x^{-1}\}$.

   (b) Find the closure of each set, and decide whether this closure is compact.

2. (a) Prove that if U and V are open subsets of $\mathbb{R}^n$, then $W = U \cap V$ is also open.

   (b) Prove that if $\{U_i : i \in I\}$ is a (possibly infinite) collection of open sets, then the union $U = \bigcup_{i \in I} U_i$ is open.

3. If $(\mathbf{x}_k)$ is a sequence in $\mathbb{R}^n$ and $\lim\limits_{k\to\infty} \mathbf{x}_k = \mathbf{x}$, prove that $\lim\limits_{k\to\infty} \|\mathbf{x}_k\| = \|\mathbf{x}\|$.

4. Suppose that $(\mathbf{x}_k)$ is a sequence in $\mathbb{R}^n$ and $\sum\limits_{k=1}^{\infty} \|x_{k+1} - x_k\| < \infty$.

   Prove that $(\mathbf{x}_k)$ is a Cauchy sequence.

   Hint: if $\varepsilon > 0$, there is an N so that $\sum\limits_{k=N}^{\infty} \|x_{k+1} - x_k\| < \varepsilon$.

5. Show that if $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are compact sets, then

$$A \times B = \{(a, b) \in \mathbb{R}^{m+n} : a \in A \text{ and } b \in B\}$$

   is compact. **Hint:** take a sequence in $A \times B$, find a subsequence based on the first coordinate, and then find a sub-subsequence based on the second coordinate.