AMath/PMath 331
Assignment 1
Due Friday September 23

1. Let $a_n = \sqrt{n^2 - n} - n$ for $n \geq 1$.
   (a) Compute $L := \lim_{n\to\infty} a_n$.
   (b) Estimate the error $|L - a_n|$, and find an integer N so that
   $$|L - a_n| < \tfrac{1}{2}10^{-6} \quad \text{for all} \quad n \geq N.$$

2. Let $a_n = \sin(\log n)$ for $n \geq 1$.
   (a) Show that there is an integer in the interval $[e^{\pi(k+\frac{1}{3})}, e^{\pi(k+\frac{2}{3})}]$ for each $k \geq 0$.
   (b) What does part (a) say about certain terms of this sequence?
   (c) Prove that the sequence (a_n) does not converge.

3. Let $S = \{x : 0 < \sin(1/x) < 1/2\}$. Find $\inf S$.

4. Let $a_1 = 0$ and $a_{n+1} = \sqrt{5 + 2a_n}$ for $n \geq 1$.
   (a) Show by induction that $a_n < a_{n+1} < 4$.
   (b) Prove that $L := \lim_{n\to\infty} a_n$ exists, and calculate it.
   (c) Show that $|L - a_{n+1}| < \frac{1}{2}|L - a_n|$. Hence find an integer N so that
   $$|L - a_n| < \tfrac{1}{2}10^{-12} \quad \text{for all} \quad n \geq N.$$

5. Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Let $L = \sup S$.
   (a) Prove that there is a sequence (s_n) consisting of points in S such that $\lim_{n\to\infty} s_n = L$.
   (b) Is it always possible to choose the s_n to be distinct points in S?