Math 245 Solutions to supplementary problems on duality

S1. (a) Compute:

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 2 \\ -.5 & 1 & -.5 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

So $w_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and $w_3 = \begin{bmatrix} -.5 & 1 & -.5 \end{bmatrix}$. Column vectors act on row vectors by multiplication on the left. The identity $A^{-1}A = I$ yields $w_i v_j = \delta_{ij}$, meaning that $\{w_1, w_2, w_3\}$ is the dual basis to $\{v_1, v_2, v_3\}$.

(b) Following the ideas from (a), let

$$C = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix} = \begin{bmatrix} \varphi_j(x^{i-1}) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 2 & \frac{9}{2} \\ \frac{1}{3} & \frac{8}{3} & 9 \end{bmatrix}.$$

This has inverse $C^{-1} = \begin{bmatrix} 3 & -5 & \frac{3}{2} \\ -\frac{3}{2} & 4 & -\frac{3}{2} \\ \frac{1}{3} & -1 & \frac{1}{2} \end{bmatrix}$. Therefore

$$I = C^{-1}C = C^{-1} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix} = \begin{bmatrix} 3 - 5x + \frac{3}{2}x^2 \\ -\frac{3}{2} + 4x - \frac{3}{2}x^2 \\ \frac{1}{3} - x + \frac{1}{2}x^2 \end{bmatrix} \begin{bmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{bmatrix}$$

Hence the dual basis is $\left\{3 - 5x + \frac{3}{2}x^2, -\frac{3}{2} + 4x - \frac{3}{2}x^2, \frac{1}{3} - x + \frac{1}{2}x^2\right\}$.

S2. (a) For any $B \in \mathcal{L}(V)$, we have

$$T^t(\tau)(B) = \tau(T(B)) = \tau(AB - BA) = 0.$$

Therefore $T^t(\tau) = 0$.

(b) Let $a_{ij} = \varphi(E_{ji})$ and define $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. If $B = \begin{bmatrix} b_{ij} \end{bmatrix}$, then

$$\begin{aligned} \tau(AB) &= \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} \varphi(E_{ki}) = \varphi\Big(\sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} E_{ki} \Big) = \varphi(B). \end{aligned}$$

S3. (a) $E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii}$ and $E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij}$ for $i \neq j$.

(b) If $\tau(A) = 0$, then $a_{11} = -\sum_{i=2}^{n} a_{ii}$. Therefore

$$\begin{aligned} A &= \sum_{i \neq j} a_{ij} E_{ij} + \sum_{i=2}^{n} a_{ii}(E_{ii} - E_{11}) \\ &= \sum_{i \neq j} a_{ij}(E_{ii}E_{ij} - E_{ij}E_{ii}) + \sum_{i=2}^{n} a_{ii}(E_{i1}E_{1i} - E_{1i}E_{i1}) \end{aligned}$$

is a sum of commutators. Conversely, every commutator is in $\ker \tau$ because $\tau(AB - BA) = \tau(AB) - \tau(AB) = 0$. So the commutators span all of $\ker \tau$.

(c) If $\ker\varphi$ contains $ST - TS$ for every $S, T \in \mathcal{L}(V)$, then by (b), $\ker\varphi \supset \ker\tau$. Now $\mathcal{L}(V) = \ker\tau + \mathbb{C}I$. If $\varphi(I) = a$, then for every $T \in \ker\tau$ and $\lambda \in \mathbb{C}$,

$$\varphi(T + \lambda I) = \lambda a = \frac{a}{n}\tau(T + \lambda I).$$

So $\varphi = \frac{a}{n}\tau$.

S4. $\ker\rho_T = \{p : p(T) = 0\} = (m_T)$. The range of ρ_T is $\mathcal{A}(T)$. So we obtain the factorization

$$\begin{array}{ccc} \mathbb{F}[x] & \xrightarrow{\ \rho_T\ } & \mathcal{L}(V) \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\iota} \\ \mathbb{F}[x]/(m_T) & \xrightarrow[\simeq]{\ \tilde{\rho}_T\ } & \mathcal{A}(T) \end{array}$$

S5. (a)

$$\begin{array}{ccc} V & \xrightarrow{\ E\ } & V \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\iota} \\ V/N & \xrightarrow[\simeq]{\ \tilde{E}\ } & M \end{array}$$

(b)

$$\begin{array}{ccc} V' & \xrightarrow{\ E^t\ } & V' \\ {\scriptstyle\iota^t}\downarrow & & \uparrow{\scriptstyle\pi^t} \\ M' & \xrightarrow[\simeq]{\ \widetilde{E^t}\ } & (V/N)' \end{array} \quad = \quad \begin{array}{ccc} V' & \xrightarrow{\ E^t\ } & V' \\ {\scriptstyle\iota^t}\downarrow & & \uparrow{\scriptstyle\pi^t} \\ V/M^\perp & \xrightarrow[\simeq]{\ \tilde{E}^t\ } & N^\perp \end{array}$$

E^t is an idempotent in $\mathcal{L}(V')$ with range $N^\perp$ and kernel $M^\perp$.