Math 245
Supplementary problems on duality

S1. (a) Let $v_1 = (1, 0, -1)^t$, $v_2 = (1, 1, 1)^t$, and $v_3 = (2, 2, 0)^t$. This is a basis for $\mathbb{R}^3$. Find the corresponding dual basis.
Hint: Find the inverse of the matrix $[v_1, v_2, v_3]$.

(b) Let V be the vector space of polynomials of degree 2. Consider the basis for V' given by:

$$\varphi_i(p) = \int_0^i p(x)\,dx \quad \text{for} \quad i = 1, 2, 3.$$

Find a basis for V dual to $\{\varphi_1, \varphi_2, \varphi_3\}$.

S2. (a) Fix $A \in \mathcal{L}(V)$. Define $T \in \mathcal{L}(\mathcal{L}(V))$ by $T(B) = AB - BA$. Let τ be the trace functional on $\mathcal{L}(V)$ given by $\tau(B) = \text{trace}(B)$. Compute $T^t(\tau)$.

(b) Let φ be a functional in $\mathcal{L}(V)'$. Show that there is a unique transformation A in $\mathcal{L}(V)$ so that $\varphi(B) = \tau(AB)$ for all $B \in \mathcal{L}(V)$. **Hint:** consider $\varphi(E_{ij})$.

S3. (a) Show that E_{ij} and $E_{ii} - E_{jj}$ are *commutators*, meaning they are of the form $AB - BA$, for $i \neq j$.

(b) Hence show that the linear span of all commutators consists of all matrices of trace zero.

(c) Hence show that if $\varphi \in \mathcal{L}(V)'$ satisfies $\varphi(AB - BA) = 0$ for all A and B in $\mathcal{L}(V)$, then F is a multiple of τ, the trace functional.

S4. Let $T \in \mathcal{L}(V)$. Let ρ_T be the linear map from $\mathbb{F}[x]$ to $\mathcal{L}(V)$ given by $\rho_T(p) = p(T)$. Explain the factorization of ρ_T through the quotient by the kernel of ρ_T to the range of ρ_T and then into $\mathcal{L}(V)$ using the language developed to study linear transformations.
(*Note: thinking of ρ_T as a linear map hides the fact that it is also multiplicative.*)

S5. Let $E \in \mathcal{L}(V)$ be an idempotent with range M and kernel N.

(a) Explain the factorization of E through the quotient by the kernel, to the range, and then back into V.

(b) Explain what happens when you take the transpose of this factorization. In particular, what is the range of E^t?