Math 245 Assignment 6
Due Friday November 30

1. A quadratic form is a real function of n variables $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ of the form
$$q(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{i} a_{ij} x_i x_j .$$
   (a) Show that there is a real *symmetric* matrix $A = A^*$ so that $q(x) = \langle Ax, x \rangle$.

   (b) Use the fact that A can be diagonalized to show that there are linear functions $L_i(x) = \sum_{j=1}^{n} b_{ij} x_j$ for $1 \le i \le n$ so that $q(x) = \sum_{i=1}^{n} \varepsilon_i L_i(x)^2$ where $\varepsilon_i \in \{1, 0, -1\}$.

   (c) Deduce that if a quadratic form takes only non-negative values, then it is the sum of squares of linear functions.

   (d) **Bonus.** Show that the polynomial $p(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ takes only non-negative values, but is not the sum of squares of polynomials. **Hint:** Use the arithmetic mean-geometric mean inequality. For the second part, restrict attention to a sum of squares of homogeneous cubics.

2. The *Schur product* of two matrices $\big[a_{ij}\big]$ and $\big[b_{ij}\big]$ is $\big[a_{ij}b_{ij}\big]$. Prove that the Schur product of two positive matrices is positive. **Hint:** first establish this for two positive rank one matrices. Then express each positive matrix as a sum of positive rank one matrices.

3. Let M and N be subspaces of a vector space V. ($M^\perp$ is the annihilator of M in V'.)

   (a) Show that $(M + N)^\perp = M^\perp \cap N^\perp$.

   (b) Show that $(M \cap N)^\perp = M^\perp + N^\perp$.

   (c) Show that the map J from $M/(M \cap N)$ to $(M + N)/N$ given by $J\big(m + (M \cap N)\big) = m + N$ is a linear isomorphism.

   (d) Provide an example to show that if V is a normed vector space, then J may not be isometric. **Hint:** this can be done with $V = (\mathbb{R}^2, \|\cdot\|_2)$.

4. Let A and B be closed convex sets in $\mathbb{R}^n$, and let C be a compact convex set.

   (a) Define $A + B = \{a + b : a \in A,\ b \in B\}$. Prove that $A + B$ is convex.

   (b) Show that $A + C = B + C$ implies that $A = B$. **Hint:** separation theorem.

   (c) **Bonus.** Show that $A + C$ is closed. (This is easy, but it is real analysis.)

5. Let V be a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let C be a closed bounded balanced convex subset of V containing 0 in its interior. Show that C is the unit ball of a norm on V.

6. Let V be a real vector space. A subset S of V is *affine* if $s_1, s_2 \in S$ implies that $ts_1 + (1 - t)s_2 \in S$ for all $t \in \mathbb{R}$. The *affine hull* of $S \subset V$ is $\mathrm{aff}(S) = \big\{ \sum_i t_i s_i : s_i \in S,\ t_i \in \mathbb{R},\ \sum_i t_i = 1 \big\}$. A set S is *affinely dependent* if some $s_0 \in S$ is in the affine hull of $S \setminus \{s_0\}$. Otherwise it is *affinely independent.*

   (a) Show that $\mathrm{aff}(S)$ is the smallest affine set containing S.

   (b) Show that $\mathrm{aff}(S)$ is a translate of a subspace.

   (c) Show that $s_0, s_1, \dots, s_k$ are affinely dependent if and only if $s_1 - s_0, \dots, s_k - s_0$ are linearly dependent.

   (d) Let $n = \dim V$. Suppose that $r \ge n + 2$ and $s_1, \dots, s_r$ are vectors in V. Show that there are disjoint sets I and J with $I \cup J = \{1, 2, \dots, r\}$ so that $C_I = \mathrm{conv}\{s_i : i \in I\}$ and $C_J = \mathrm{conv}\{s_i : i \in J\}$ intersect. **Hint:** use (c) to get a non-trivial affine relation; split into positive and negative terms.

   (e) **Bonus.** *Helly's Theorem*: let $C_1, \dots, C_m$ be convex sets in V such that any $n + 1$ of them have non-empty intersection. Prove that the whole collection has non-empty intersection. **Hint:** induction on $m \ge n + 1$. Pick s_j in $\bigcap_{i \ne j} C_i$ for $1 \le j \le m$. Apply (d).