Reminder: Midterm is Tuesday November 6, 4:30-6:20 in MC 4064.

Math 245
Assignment 5
Due Wednesday November 14

1. (a) Apply the Gram-Schmidt process to

$$v_1 = \begin{bmatrix}1\\2\\2\\4\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}0\\1\\0\\2\end{bmatrix}, \quad v_4 == \begin{bmatrix}4\\0\\-4\\1\end{bmatrix}.$$

   (b) Find the matrix (with respect to the standard basis) of the orthogonal projection of $\mathbb{R}^4$ onto $\operatorname{span}\{v_1, v_2\}$.

2. Find an orthonormal basis which diagonalizes $T = \begin{bmatrix} 0 & 20 & -3 \\ 20 & 0 & -3 \\ -3 & -3 & 13 \end{bmatrix}$, and find a unitary U so that U^*TU is diagonal.

3. Let V be a vector space over $\mathbb{C}$ with a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle$. i.e. $\langle v, v \rangle \geq 0$.

   (a) Show that $N := \{v : \langle v, v \rangle = 0\}$ is a subspace.
   Hint: first show that $N = \{v : \langle v, u \rangle = 0 \text{ for all } u \in V\}$.

   (b) Put a relation on V by $u \equiv v$ if and only if $u - v \in N$. Prove that this is an equivalence relation.

   (c) Let $W = V/N$ denote the set of equivalence classes. Show that this is a vector space with the operations $\alpha[v] = [\alpha v]$ and $[u] + [v] = [u + v]$. That is, show that these operations are well defined, and that W satisfies the properties of a vector space.

   (d) Define an inner product on W by $\langle [u], [v] \rangle = \langle u, v \rangle$. Prove that this is well defined. Then show that it is a positive definite sesquilinear form on W.

4. (a) Hence show that $T \in \mathcal{L}(V)$ on a complex inner product space V is *positive* (i.e. self-adjoint with non-negative eigenvalues) if and only if $\langle Tx, x \rangle \geq 0$ for all $x \in V$.
   Hint: first show that T is self-adjoint by using this analogue of Assignment 3, 5(a):

$$\langle Tx, y \rangle = \tfrac{1}{4}\big(\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle + i\langle T(x+iy), x+iy \rangle - i\langle T(x-iy), x-iy \rangle\big).$$

   (b) If T is any matrix in $\mathcal{L}(V)$, show that T^*T is positive.

   (c) Find a matrix T acting on $\mathbb{R}^2$ which is not self-adjoint, but $\langle Tx, x \rangle = 0$ for all $x \in \mathbb{R}^2$.

5. Show that two normal matrices N and M can be diagonalized by the same orthonormal basis (*simultaneous diagonalization*) if and only if $NM = MN$.
   Hint: ($\Longleftarrow$) start with one, say N, already diagonalized, and compute $NM = MN$ to find conditions on the matrix form of M.

6. **Bonus.** Show that a positive matrix A has a unique positive square root.
   Hint: Diagonalize A and find the 'obvious' square root B. Notice that there must be a polynomial p so that $B = p(A)$. Hence show that if C is another square root, then $BC = CB$. Apply problem 5.