Math 245
Assignment 4
Due Wednesday October 31

1. Let $T \in \mathcal{L}(V)$ where V is a vector space over an algebraically closed field.
   (a) Show that $\mathcal{A}'(T) := \{A \in \mathcal{L}(V) : AT = TA\}$ is an algebra containing $\mathcal{A}(T)$, and every $A \in \mathcal{A}'(T)$ commutes with every $S \in \mathcal{A}(T)$.
   (b) Compute $\mathcal{A}'(J_n)$.
   (c) Show that if T has a cyclic vector, then $\mathcal{A}'(T) = \mathcal{A}(T)$.
   Hint: any $A \in \mathcal{A}'(T)$ commutes with the projections E_i onto $\ker((T - \lambda_i I)^{d_i})$.
   (d) Show that if $\mathcal{A}'(T) = \mathcal{A}(T)$, then T has a cyclic vector.

2. Consider the linear recursion $x_{k+4} - 8x_{k+3} + 24x_{k+2} - 32x_{k+1} + 16x_k = 0$.
   (a) Set this up as a linear algebra problem as in class. Find the Jordan form for the matrix T.
   (b) Find an explicit basis that puts it in Jordan form.
   (c) Find a formula for x_k given $(x_0, x_1, x_2, x_3) = (1, 2, 3, 4)$.

3. Consider a (fictional) chemical in solution in a 3-chambered bottle. Because of the different conditions in the three chambers, molecules of the chemical pass across the membrane from one chamber to another with different probabilities. In one minute, the probability that a molecule in chamber C_j moves to chamber C_i is p_{ij} where $A = \begin{bmatrix} p_{ij} \end{bmatrix} = \begin{bmatrix} .4 & .2 & 0 \\ .3 & .5 & .2 \\ .3 & .3 & .8 \end{bmatrix}$. Find the limit state.

4. (a) Suppose that a_k are real numbers such that $\sum_{k=1}^{n} a_k = 1$. Prove that $\sum_{k=1}^{n} 2^k a_k^2 > 1$.
   (b) Find the minimum of $\sum_{k=1}^{n} 2^k a_k^2$ as $\{a_k\}$ runs over all real solutions of $\sum_{k=1}^{n} a_k = 1$.

5. (a) If V is a complex inner product space, show that
   $$\langle x, y \rangle = \tfrac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\big).$$
   (b) If V is a real inner product space, show that if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, then x and y are orthogonal. What can you deduce from this identity in a complex inner product space?

6. **Bonus.** Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times m$ matrix with complex entries. Define
   $$R = \max_{1 \le i \le n} \sum_{j=1}^{m} |a_{ij}| \quad \text{and} \quad C = \max_{1 \le j \le m} \sum_{i=1}^{n} |a_{ij}|.$$
   For $x = (x_1, \dots, x_m)^t \in \mathbb{C}^m$ and $y = (y_1, \dots, y_n)^t \in \mathbb{C}^n$, prove that
   $$|\langle Ax, y \rangle| \le \sqrt{RC}\, \|x\|_2\, \|y\|_2.$$