Math 245
Assignment 3
Due Friday October 19

1. (a) Let $A, B \in \mathcal{L}(V)$. Prove that $\text{nul}(AB) \leq \text{nul}(A) + \text{nul}(B)$.
   (b) Prove that T is diagonalizable if and only if the minimal polynomial m_T splits into linear factors and has no repeated roots.

2. Find the possible Jordan forms that T could have given the following information.
   (a) $p_T(x) = (x-4)^4(x+1)^2$ and $m_T(x) = (x-4)^3(x+1)^2$.
   (b) $p_T(x) = (x-4)^4(x+1)^2$ and $m_T(x) = (x-4)^2(x+1)$.

3. (a) Find the Jordan form for
$$T = \begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 0 & 9 & 0 & 15 \\ 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & -6 & 0 & -10 \end{bmatrix}.$$
   (b) Find an explicit S so that STS^{-1} is in Jordan form.
   (c) Hence find a square root for T.

4. Let $T = J(0,2) \oplus J(2,3) \oplus J(3,1)$. Find a polynomial p so that $p(T) = A$ where
$$T = \left[\begin{array}{cc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right] \quad \text{and} \quad A = \left[\begin{array}{cc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. Suppose that x is a *cyclic vector* for T, meaning that $\{x, Tx, T^2x, \dots\}$ span V.
   (a) Show that $\{x, Tx, \dots, T^{n-1}x\}$ is a basis.
   (b) Prove that $m_T = p_T$.

6. **Bonus**. Let $T \in \mathcal{L}(V)$ over an arbitrary field $\mathbb{F}$.
   (a) Show that T has a block upper triangular form so that the diagonal entries T_i, $1 \leq i \leq k$, each have a cyclic vector.
   (b) Let $p_i(x) = p_{T_i}(x)$. Show that each p_i divides m_T. Hence deduce that every irreducible factor of p_T is a factor of m_T.