Math 245
Assignment 2
Due Friday October 5

1. Consider the linear recursion $a_{n+3} = 3a_{n+2} - a_{n+1} - a_n$ for $n \geq 0$.
   (a) If $a_0 = a_1 = 1$ and $a_2 = 2$, find a formula for a_n.
   (b) Find $\lim_{n\to\infty} a_n^{1/n}$.
   What happens if $a_0 = a_1 = a_2 = 1$?
   What happens if $a_0 = a_1 = 1$ and $a_2 = 1 + \varepsilon$ for some very small $\varepsilon \neq 0$?

2. Let $E \in \mathcal{L}(V)$ such that E is idempotent, i.e. $E^2 = E$.
   (a) Let $R = \operatorname{Ran} E$ and $K = \ker E$. Show that $V = R \oplus K$.
   (b) Express E as a 2×2 matrix with *explicit* matrix entries with respect to this decomposition.
   (c) Show that E is diagonalizable.
   (d) Suppose that v_1, v_2, v_3, v_4 is a basis for V and $R = \operatorname{span}\{v_1, v_2\}$. Find the matrices of *all* idempotents E with range R w.r.t. this basis.

3. (a) Suppose that $A \in \mathcal{L}(V)$ has $n = \dim V$ distinct eigenvalues. Suppose that $B \in \mathcal{L}(V)$ commutes with A (i.e. $AB = BA$). Show that there is a polynomial p so that $B = p(A)$.
   (b) Show that this conclusion fails for the matrix
   $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. Let V be a finite dimensional vector space over $\mathbb{C}$. Suppose that $A, B \in \mathcal{L}(V)$ commute (i.e. $AB = BA$).
   (a) Show that if $p \in \mathbb{C}[x]$ is any polynomial, then $\ker(p(A))$ is invariant for B.
   (b) Show that A and B have a common eigenvector.
   Hint: Take $\alpha \in \sigma(A)$ and let B_1 be the restriction of B to $W = \ker(A - \alpha I)$. Find an eigenvector for B_1.
   (c) Show that A and B can be simultaneously triangularized. i.e. find a basis $\mathcal{B}$ so that both $[A]_{\mathcal{B}}$ and $[B]_{\mathcal{B}}$ are triangular. **Hint:** use induction on n as in the proof of triangularizability of a single map.
   (d) **Bonus.** Generalize this to a commuting family of transformations on V.