Math 245
Assignment 1
Due Friday September 28

In this assignment, V is always a finite dimensional vector space over a field $\mathbb{F}$.

1. (a) Find all eigenvalues and eigenspaces for the following matrices ($\mathbb{F} = \mathbb{C}$):

   (i) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 7 & -5 & -5 \\ -5 & 2 & 0 \\ -5 & 0 & 2 \end{bmatrix}$ (iii) $C = \begin{bmatrix} -3 & 3 & 0 \\ -5 & 1 & 4 \\ -4 & 2 & 2 \end{bmatrix}$.

   (b) Which of these matrices can be diagonalized? Which can be diagonalized if $\mathbb{F} = \mathbb{R}$?

2. Suppose that $V_1, \dots, V_k$ are subspaces such that $V = \sum_{i=1}^{k} V_i$. Prove that this is a direct sum if and only if

$$\sum_{i=1}^{k} \dim V_i = \dim V.$$

3. Suppose that $V = V_1 \oplus V_2$.

   (a) Show that $T \in \mathcal{L}(V)$ can be written as $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ where $T_{ij} \in \mathcal{L}(V_j, V_i)$ for $1 \le i, j \le 2$. **Hint:** choose a basis for V_1 and V_2, combine to get a basis for V, and decompose the matrix for T in this basis.

   (b) Suppose that $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$. Show that $p_T(x) = p_{T_{11}}(x) p_{T_{22}}(x)$.

4. Let $A, B \in \mathcal{L}(V)$.

   (a) Show that AB is singular if and only if BA is singular.

   (b) If $0 \ne \lambda \in \mathbb{F}$, and $\lambda \notin \sigma(BA)$, calculate

$$\big(A(BA - \lambda I)^{-1}B - I\big)(AB - \lambda I).$$

   (c) Hence show that AB and BA have the same spectrum.

5. **Bonus.** Let $T \in \mathcal{L}(V)$. Define $L \in \mathcal{L}(\mathcal{L}(V))$ by $L(A) = TA$ for $A \in \mathcal{L}(V)$. Express p_L in terms of p_T.