DILATION THEORY: SOLUTIONS TO ASSIGNMENT 1

THANKS TO MICHAEL HARTZ AND MATTHEW WIERSMA

1. (a) If V is a pure isometry, then by the Wold decomposition, it is a direct sum of unilateral shifts. So $M = \ker V^*$ is a wandersing space, and $\mathcal{K} = \sum_{n\geq 0}^{\oplus} V^n M$. For any vector x in the algebraic sum $\sum_{n\geq 0} V^n M$, there is an n so that $V^{*n}x = 0$. By continuity, $\lim\limits_{n\to\infty} V^{*n}x = 0$ for all $x \in \mathcal{K}$. In particular, for $h \in \mathcal{H}$, $\lim\limits_{n\to\infty} T^{*n}h = \lim\limits_{n\to\infty} V^{*n}h = 0$.

(b) Let $\mathcal{L} = \mathcal{K}_- \oplus \mathcal{H} \oplus \mathcal{K}_+$ be an orthogonal decomposition of $\mathcal{L}$ such that U has a matrix representation

$$U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix}.$$

Now $\mathcal{M}_+ = (\mathcal{H} \vee U\mathcal{H}) \ominus \mathcal{H}$ is contained in $\mathcal{K}_+$, which is invariant for U. So for $n \geq 0$, $U^n \mathcal{M}_+ \subset \mathcal{K}_+$ and thus is orthogonal to $\mathcal{H}$. Applying U to both spaces, we get that for $n \geq 1$, $U^n\mathcal{M}_+$ is orthogonal to $U\mathcal{H}$. So for $n \geq 1$, $U^n\mathcal{M}_+$ is orthogonal to $\overline{\mathcal{H} + U\mathcal{H}}$ which contains $\mathcal{M}_+$. Apply U^m, for $m \in \mathbb{Z}$, to both sides to get $U^{n+m}\mathcal{M}_+ \perp U^m\mathcal{M}_+$ for all $n + m > m$ in $\mathbb{Z}$.

Choose an orthonormal basis $\{e_0^\alpha : \alpha \in A\}$ for $\mathcal{M}_+$. Then $\{e_n^\alpha = U^n e_0^\alpha : \alpha \in A,\, n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{L}_+$. The subspace $\mathcal{H}_\alpha = \operatorname{span}\{e_n^\alpha : n \in \mathbb{Z}\}$ is U-invariant, and $U|_{\mathcal{H}_\alpha}$ is unitarily equivalent to the bilateral shift for each α; and $\mathcal{L}_+ = \sum_{\alpha\in A}^{\oplus} \mathcal{H}_\alpha$.

The argument for the restriction to $\mathcal{L}_-$ is similar.

Note: The space $L_+ \vee \mathcal{L}_-$ *is reducing for* U. *You cannot say that the restriction is a bilateral shift. However the spectral measure on each subspace is absolutely continuous with respect to Lebesgue measure, and the same follows for the restriction to the sum. It could, for example, end up as the direct sum of a bilateral shift and multiplication by* z *on* $L^2(A)$ *for some measurable subset* $A \subset \mathbb{T}$. *If either* $\dim \mathcal{M}_\pm = \infty$, *then the restriction to the sum is unitarily equivalent to countably many copies of the bilateral shift.*

(c) Consider the subspace $\mathcal{L}_0 = \bigvee_{n\geq 0} U^n\mathcal{M}_+ \vee \bigvee_{n\geq 0} U^{-n}\mathcal{M}_- \vee \mathcal{H}$. Observe that

$$U\mathcal{H} \subset \mathcal{H} + \mathcal{M}_+ \quad \text{and} \quad U\mathcal{M}_- \subset U(\mathcal{H} \vee U^*\mathcal{H}) = \mathcal{H} \vee U\mathcal{H} = \mathcal{H} + \mathcal{M}_+,$$

and similarly

$$U^*\mathcal{H} \subset \mathcal{H} + M_- \quad \text{and} \quad U^*\mathcal{M}_+ \subset U^*(\mathcal{H} \oplus U\mathcal{H}) = \mathcal{H} \vee U^*\mathcal{H} = \mathcal{H} + \mathcal{M}_+.$$

Thus $\mathcal{L}_0$ is invariant for both U and U^*. By minimality of the dilation U, we have $\mathcal{L}_0 = \mathcal{L}$.

Now $\bigvee_{n\geq 0} U^n\mathcal{M}_+ \subset \mathcal{K}_+$ and $\bigvee_{n\geq 0} U^{-n}\mathcal{M}_- \subset \mathcal{K}_-$. It follows that these must be equalities. Hence $\mathcal{L}_+ \vee \mathcal{L}_- \supset \mathcal{K}_+ + \mathcal{K}_- = \mathcal{H}^\perp$. Therefore $\mathcal{N} = (\mathcal{L}_+ \vee \mathcal{L}_-)^\perp \subset \mathcal{H}$. The complement of a reducing subspace is reducing. So $T|_{\mathcal{N}} = P_{\mathcal{H}}U|_{\mathcal{N}} = U|_{\mathcal{N}}$ is unitary.

Note: A contraction with no reducing subspace on which it is a unitary is called completely non-unitary (cnu). *It follows that the minimal unitary dilation of a cnu contraction is absolutely continuous with respect to Lebesgue measure.*

2. *Note: it is necessary to assume that* $\mathcal{K}$ *is separable. It is easy to see that the smallest invariant subspace for a countable family of operators containing a countable set of vectors is separable.*

By the Wold decomposition, V_1 is a direct sum of unilateral shifts. There are two cases depending on whether there are finitely many or countably many.

Case 1. $V_1 \simeq S^{(n)}$. Now $e_0 \in \mathcal{M} = \ker V_1^*$. Rename it $e_0 = e_0^1$ and extend it to a basis $\{e_0^i : 1 \le i \le n\}$ for $\mathcal{M}$. Then $\{e_k^i = V_1^k e_0^i : 1 \le i \le n,\ k \ge 0\}$ forms an orthonormal basis for $\mathcal{K}$. Define

$$V_2 e_k^i = \begin{cases} e_k^{i+1} & \text{for} \quad 1 \le i \le n-1 \\ e_{k+1}^1 & \text{for} \quad i = n \end{cases}.$$

Then an easy calculation shows that $V_2^n = V_1$; so they commute/ Moreover V_2 is a unilateral shift with $\ker V_2^* = \mathbb{C}e_0$. So V_2 is the minimal isometric dilation of $A_2 = [0]$. Therefore the pair $\{V_1, V_2\}$ is a minimal commuting isometric dilation of $\{A_1, A_2\}$.

Case 2. $V_1 \simeq S^{(\infty)}$. Again set $e_0 = e^1 + 0$ and extend it to a basis $\{e_0^i : i \ge 1\}$ for $\mathcal{M} = \ker V_1^*$; and $\{e_k^i = V_1^k e_0^i : i \ge 1,\ k \ge 0\}$ forms an orthonormal basis for $\mathcal{K}$. Define $V_2 e_k^i = e_k^{i+1}$ for all i, k. It is easy to verify that V_2 is an isometry with $e_0 \in \ker V_2^*$, and that it commutes with V_1. Moreover $V_1^k V_2^{i-1} e_0^1 = e_k^i$ for all $k \ge 0$ and $i \ge 1$. So $\mathcal{K}$ is the minimal invariant subspace for $\{V_1, V_2\}$ containing e_0.

3. (a) Since $\operatorname{spr}(T) < 1$, we can find an integer N and $0 \le r < 1$ so that $\|T^n\|^{1/n} \le r$ for $n \ge N$. Thus $\|T^{*n}T^n\| \le r^{2n}$ for $n \ge N$ and, hence, $\sum_{n=0}^\infty T^{*n}T^n$ converges absolutely. Since $T^{*n}T^n = (T^n)^*(T^n) \ge 0$ for every n and the positive elements of a C^*-algebra form a closed cone, $\sum_{n=1}^\infty T^{*n}T^n$ is positive. Therefore $A = I + \sum_{n=1}^\infty T^{*n}T^n \ge I$. It follows that $\sigma(A) \subset [1, \infty)$, and so A is invertible. Therefore, $I = A^{-1/2} A A^{-1/2} \ge A^{-1/2} I A^{-1/2} = A^{-1}$.

(b) Note that $A = I + T^*AT$. Thus, by (a)

$$\|A^{1/2} T A^{-1/2}\|^2 = \|A^{-1/2} T^* A T A^{-1/2}\| = \|A^{-1/2}(A - I)A^{-1/2}\| = \|I - A^{-1}\| \le 1.$$

(c) The basic structure of compact operators tells us the following: The spectrum of K is contained in $\overline{\mathbb{D}}$ and 0 is the only cluster point. So $\sigma(K) \cap \mathbb{T} = C$ is a finite set $C = \{\lambda_1, \dots, \lambda_n\}$. Moreover for each i, there is an integer N_i so that

$$\mathcal{N}_i = \ker(K - \lambda_i I)^N_i = \ker(K - \lambda_i I)^n \quad \text{for all} \quad n \ge N_i.$$

This subspace is finite dimensional. Then $\mathcal{R}_i = \operatorname{Ran}(K - \lambda_i I)^N_i = \operatorname{Ran}(K - \lambda_i I)^n$ for all $n \ge N_i$; and this is a closed subspace. Moreover, both are invariant for K, $\sigma(K|_{\mathcal{N}_i}) = \{\lambda_i\}$ and $\sigma(K|_{\mathcal{R}_i}) = \sigma(K) \setminus \{\lambda_i\}$ and

$$\mathcal{N}_i + \mathcal{R}_i = \mathcal{H} \quad \text{and} \quad \mathcal{N}_i \cap \mathcal{R}_i = \{0\}.$$

Observe that K is similar to $K|_{\mathcal{N}_i} \oplus K|_{\mathcal{R}_i}$ via the isomorphism

$$S : \mathcal{N}_i \oplus \mathcal{R}_i \to \mathcal{H} \quad \text{by} \quad S(x, y) = x + y.$$

If we repeat this procedure for $1 \le i \le n$, we obtain a decomposition $K \sim K_0 \oplus T$ where $\sigma(K_0) = \sigma(K) \setminus C \subset \mathbb{D}$ and $\sigma(T) = C \subset \mathbb{T}$, and T acts on $\sum_{i=1}^n \oplus \mathcal{N}_i$, which is finite dimensional.

(d) Suppose that K is similar to a contraction. Then K is completely polynomially bounded and, in particular is power bounded.

Conversely suppose that K is power bounded. Then $\sup_{n \ge 0} \|K^n\| < \infty$, and therefore

$$\operatorname{spr}(K) = \lim_{n \to \infty} \|K^n\|^{1/n} \le 1.$$

Use the decomposition from (c). Then $\operatorname{spr}(K_0) < 1$, $\sigma(T) \subset \mathbb{T}$ and T is power bounded. By Rota's Theorem (b), K_0 is similar to a contraction. The matrix T is similar to its Jordan form. So $T|_{\mathcal{N}_i} \sim J(\lambda_i, k_1) \oplus \cdots \oplus J(\lambda_i, k_s)$. I claim that each $k_j = 1$. Indeed, if

$J(\lambda_i, k)$ is a Jordan block of size bigger $k > 1$, then expand $J(\lambda_i, k)^n = (\lambda_i I + J(0,k))^n$ using the binomial theorem to get

$$J(\lambda_i, k)^n = \lambda_i^n I + n\lambda_i^{n-1} J(0,k) + \binom{n}{2} \lambda_i^{n-2} J(0,k)^2 + \cdots = \begin{bmatrix} \lambda_i^n & n\lambda_i^{n-1} & \dots \\ 0 & \lambda_i^n & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}.$$

Therefore $\|J(\lambda_i, k)^n\| \geq |n\lambda_i^{n-1}| = n$, which is not power bounded. Consequently all block are size 1, and T is similar to a diagonal matrix with diagonal entries of modulus one. This is a contraction.

4. (a) By hypothesis,

$$A^*A + C^*C = \begin{bmatrix} A \\ C \end{bmatrix}^* \begin{bmatrix} A \\ C \end{bmatrix} \leq 1.$$

In particular, $A^*A \leq 1$, so that $D_A = (1 - A^*A)^{1/2}$ makes sense. Define an operator L on the range of D_A by

$$L(D_A x) = Cx \quad \text{for} \quad x \in \mathcal{H}.$$

Note that

$$\|Cx\|^2 = \langle C^*Cx, x\rangle \leq \langle (1 - A^*A)x, x\rangle = \|D_A x\|^2.$$

Hence L is a well-defined contraction on $\operatorname{Ran} D_A$. Extend L by continuity to $\overline{\operatorname{Ran} D_A}$ and set $L = 0$ on $(\operatorname{Ran} D_A)^\perp$. We obtain a contraction on $\mathcal{H}$ which clearly satisfies $C = LD_A$.

If $\| \begin{bmatrix} A & B \end{bmatrix} \| \leq 1$, then we can apply the first part to the adjoint of this row operator to obtain a contraction K^* such that $B^* = K^* D_{A^*}$. Hence, $B = D_{A^*} K$.

(b) By part (a), we can find contractions K and L so that $B = D_{A^*} K$ and $C = LD_A$. Take $X = -KA^*L$. Then

$$\begin{bmatrix} A & B \\ C & X \end{bmatrix} = \begin{bmatrix} A & D_{A^*}L \\ KD_A & -KA^*L \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & L \end{bmatrix}.$$

To show that this is a contraction, it is enough to show that

$$\left\| \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix} \right\| \leq 1.$$

We ccompute

$$\begin{aligned} \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}^* \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix} &= \begin{bmatrix} A^*A + D_A^2 & A^*D_{A^*} - D_A A^* \\ D_A^* A - AD_A & D_{A^*}^2 + AA^* \end{bmatrix} \\ &= \begin{bmatrix} I & (D_{A^*}A - AD_A)^* \\ D_{A^*}A - AD_A & I \end{bmatrix}. \end{aligned}$$

Note that $(I - AA^*)A = A(I - A^*A)$, and hence $(I - AA^*)^n A = A(I - A^*A)^n$ for every $n \in \mathcal{N}$. It follows that $p(I - AA^*)\, A = A\, p(I - A^*A)$ for every polynomial p. Approximating the square root function on $[0,1]$ uniformly by polynomials, we get that $D_{A^*}A - AD_A = 0$. Therefore, $\begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}$ is unitary, and so has norm 1.

5. (a) Consider the $n \times n$ matrix

$$S_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Note that S_n is a contraction and nilpotent of order n. Let

$$p(z) = a_0 + a_1 z + \dots \; a_{n-1} z^{n-1} + z^n q(z) \qquad \text{for } q \in \mathbb{C}[z].$$

Then $A_n = p(S_n)$. Therefore by von Neumann's inequality,

$$\|A_n\| \le \|p(z)\|_\infty = \|a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + z^n q(z)\|_\infty.$$

Now take the infimum over all $q \in \mathbb{C}[z]$.

(b) Note that permuting the columns of a matrix amounts to composition with a unitary, and so does not change its norm. Thus, Parrot's result implies that if B is an $n \times n$ matrix, A is a $n \times 1$ matrix and C is a $1 \times n$ matrix such that $\begin{bmatrix} A & B \end{bmatrix}$ and $\begin{bmatrix} B \\ C \end{bmatrix}$ are contractions, then there exits a scalar x such that $\begin{bmatrix} A & B \\ x & C \end{bmatrix}$ is a contraction. Apply this observation to

$$\left[\begin{array}{c|cccccc} a_0 & 0 & 0 & \dots & 0 & 0 \\ a_1 & a_0 & 0 & \dots & 0 & 0 \\ a_2 & a_1 & a_0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ a_{n-1} & a_{n-2} & a_{n-1} & \dots & a_0 & 0 \\ \hline x & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{array}\right].$$

There is a solution $x = a_n$ that makes this a contraction. Applying this procedure recursively, we obtain a sequence $a_n, a_{n+1}, \ldots$ of complex numbers so that the matrices $A_{n+1}, A_{n+2}, \ldots$ all have norm 1.

Note: A different solution is provided by the Commutant Lifting Theorem. The unilateral shift S is the isometric coextension of S_n. The matrix A_n commutes with S_n. So by CLT, there is a coextension A acting on ℓ^2 which commutes with S with $\|A\| = 1$. It is an easy exercise to show that any operator commuting with S is lower triangular and the coefficients satisfy $a_{ij} = a_{i-j,0}$. These are the desired coefficients.

(c) Clearly $|a_n| \le 1$ for all n, so the power series $h(z) = \sum_{n=0}^\infty a_n z^n$ has radius of convergence at least 1, and hence defines a holomorphic function on $\mathbb{D}$. For $0 < r < 1$, let $h_r(z) = h(rz) = \sum_{n=0}^\infty a_n r^n z^n$. This series converges absolutely and uniformly on $\overline{\mathbb{D}}$ to a function in the disk algebra $A(\mathbb{D})$. If S_n is the truncated shift as in part (a), then $h_r(S_n)$ converges in norm to A_n as $r \to 1$. Thus, von Neumann's inequality for disk algebra functions yields the estimate

$$1 = \|A_n\| = \lim_{r \to 1} \|h_r(S_n)\| \le \sup_{0<r<1} \|h_r\|_\infty = \sup_{|z|<1} |h(z)|.$$

For the reverse inequality, identify $\mathbb{C}^n$ with a subspace of $\ell^2(\mathbb{N})$ in the obvious way for each $n \in \mathbb{N}$. Then the operators A_n, A_{n+1}, …can be regarded as contractions on ℓ^2

which co-extend each other. Thus there is a contraction A on ℓ^2 with infinite matrix

$$A = \begin{bmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_0 & 0 & \ddots \\ a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with coefficients $a_{i,j} = a_{i-j}$. For $z \in \mathbb{D}$, consider the unit vector

$$k_z = (1-|z|^2)^{1/2}(1, \bar{z}, \bar{z}^2, \bar{z}^3, \dots)^t.$$

Observe that

$$\begin{aligned} 1 = \|A\| \, \|k_z\|^2 &\geq |\langle Ak_z, k_z\rangle| \\ &= (1-|z|^2)\Big|\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} a_{i,j}\bar{z}^j z^i\Big| \\ &= (1-|z|^2)\Big|\sum_{n=0}^{\infty} a_n z^n \sum_{j=0}^{\infty} |z^j|^2\Big| \\ &= \Big|\sum_{n=0}^{\infty} a_n z^n\Big| = |h(z)|. \end{aligned}$$

Therefore $\sup_{z\in\mathbb{D}} |h(z)| \leq 1$.