DILATION THEORY: SOLUTIONS TO ASSIGNMENT 1

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Problem 1

(a) For $a = (a_n) \in \ell_1(\mathbb{Z})$, define

$$f_a(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad (z \in \mathbb{T}).$$

Note that the series converges absolutely and uniformly on $\mathbb{T}$, so f_a is continuous. Moreover, a_n is the nth Fourier coefficient of f_a. Therefore, we obtain a map

$$\Phi : \ell_1(\mathbb{Z}) \to \mathbb{A}(\mathbb{T}), \quad a \mapsto f_a,$$

which is obviously linear and an isometric isomorphism by definition of $\mathbb{A}(\mathbb{T})$. In particular, $\mathbb{A}(\mathbb{T})$ is a Banach space. Since $\ell_1(\mathbb{Z})$ is a Banach algebra under involution, it suffices to show that Φ is multiplicative. To this end, let $a = (a_n)$ and $b = (b_n)$ be sequences in $\ell_1(\mathbb{Z})$. Then for all $z \in \mathbb{T}$, we have

$$\begin{aligned}\Phi(a * b)(z) &= \Phi\Big(\Big(\sum_{k=-\infty}^{\infty} a_k b_{n-k}\Big)_n\Big)(z) = \sum_{n=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{\infty} a_k b_{n-k}\Big) z^n \\ &= \Big(\sum_{n=-\infty}^{\infty} a_k z^k\Big)\Big(\sum_{n=-\infty}^{\infty} b_n z^n\Big) \\ &= \Phi(a)(z)\Phi(b)(z),\end{aligned}$$

where the Cauchy product is justified due to absolute convergence of the sums. Consequently, $\ell_1(\mathbb{Z})$ and $\mathbb{A}(\mathbb{T})$ are isometrically isomorphic as Banach algebras.

(b) We claim that the maximal ideal space $\mathcal{M}_{\mathbb{A}}$ is homeomorphic to the unit circle $\mathbb{T}$. Suppose that $\varphi \in \mathcal{M}_{\mathbb{A}}$ and let $\lambda = \varphi(z)$. Since characters are contractive, and since $\|z\|_{\mathbb{A}} = 1$, we have $|\lambda| \le 1$. On the other hand, we observe that $\|z^{-1}\| = 1$, so that $|\varphi(z^{-1})| \le 1$, and hence

$$1 = |\varphi(zz^{-1})| = |\lambda|\,|\varphi(z^{-1})| \le |\lambda| \le 1.$$

Therefore, $|\lambda| = 1$, so that we may define a map

$$\Phi : \mathcal{M}_{\mathbb{A}} \to \mathbb{T}, \quad \varphi \mapsto \varphi(z).$$

By definition of the weak* topology, this map is continuous. Moreover, since $\mathbb{A}$ is a function algebra on $\mathbb{T}$, any point $\lambda \in \mathbb{T}$ gives rise to an evaluation functional $\delta_\lambda \in \mathcal{M}_{\mathbb{A}}$. Clearly, $\Phi(\delta_\lambda) = \lambda$, so that Φ is surjective.

We finish the proof by showing that Φ is injective. This is enough, because every continuous bijection between compact Hausdorff spaces is a homeomorphism. So let φ and ψ be characters on $\mathbb{A}$ satisfying $\varphi(z) = \psi(z)$. It follows that

$$\varphi(z^k) = \psi(z^k) \quad \text{for all } k \ge 0.$$

From this identity and from

$$\varphi(z^k)\varphi(z^{-k}) = \varphi(1) = 1 = \psi(1) = \psi(z^k)\psi(z^{-k}),$$

we deduce that the identity also holds for negative exponents k, so that ψ and φ agree on all trigonometric polynomials. Since these form a dense subspace of $\mathbb{A}$, we have $\varphi = \psi$, as desired.

(c) If $f \in \mathbb{A}$ has no zeros, then by part (b), we have

$$0 \notin \{\varphi(f) : \varphi \in \mathcal{M}_{\mathbb{A}}\}.$$

Gelfand theory tells us this set is the spectrum of f in $\mathbb{A}$, so that f is invertible in $\mathbb{A}$. Since $\mathbb{A}$ is a function algebra, this inverse is necessarily $1/f$, thus $1/f \in \mathbb{A}$.

PROBLEM 2

(a) The assumption on f implies in particular that $\varphi \circ f$ is continuous on the compact set D for all $\varphi \in \mathcal{X}'$. Thus, the set $f(D)$ is weakly bounded in $\mathcal{X}$, so by the uniform boundedness principle, it is norm-bounded.

(b) Let $\varphi \in \mathcal{X}'$ with $\|\varphi\| \leq 1$. Cauchy's integral formula, applied to the analytic function $\varphi \circ f$, yields for $0 < |h| < r$ the identity

$$\begin{aligned}\frac{1}{h}\big(\varphi(f(z_0+h)) - \varphi(f(z_o))\big) &= \frac{1}{h}\frac{1}{2\pi i}\int_{\partial D} \frac{(\varphi \circ f)(\zeta)}{\zeta - (z_0+h)} - \frac{(\varphi \circ f)(\zeta)}{\zeta - z_0}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial D} \frac{(\varphi \circ f)(\zeta)}{(\zeta - (z_0+h))(\zeta - z_0)} d\zeta.\end{aligned}$$

On the other hand, Cauchy's integral formula for the first derivate shows that

$$(\varphi \circ f)'(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{(\varphi \circ f)(\zeta)}{(\zeta - z_0)^2}\, d\zeta.$$

By (a), there is constant $M \geq 0$ such that $\|f(z)\| \leq M$ for all $z \in D$, so that combining the last two identities, we obtain for $0 < h < r/2$ the estimate

$$\begin{aligned}&\Big|\frac{1}{h}\big(\varphi(f(z_0+h)) - \varphi(f(z_o))\big) - (\varphi \circ f)'(z_0)\Big| \\ \leq& r \sup_{\zeta \in \partial D}\Big|\frac{(\varphi \circ f)(\zeta)}{(\zeta - (z_0+h))(\zeta - z_0)} - \frac{(\varphi \circ f)(\zeta)}{(\zeta - z_0)^2}\Big| \\ \leq& rM \sup_{\zeta \in \partial D}\Big|\frac{h}{(\zeta - (z_0+h))(\zeta - z_0)^2}\Big| \leq \frac{2M}{r^2}|h|,\end{aligned}$$

and the last quantity, which does not involve φ, clearly tends to 0 as h goes to 0.

(c) For $0 < |h| < r/2$, let

$$d_h = \frac{f(z_0+h) - f(z)}{h}.$$

Part (b) implies that the net $(\varphi(d_h))_{0<|h|<r}$ is Cauchy uniformly for $\varphi \in B_{\mathcal{X}'}$. An application of the Hahn-Banach theorem therefore shows that the net $(d_h)_{0<|h|<r/2}$ is Cauchy in the Banach space $\mathcal{X}$ and therefore convergent. By definition, this means that f is differentiable at z_0. A fortiori, it is continuous and thus, by the theory of vector valued Riemann integrals, Riemann integrable.

(d) Let $\mathcal{C}$ be a rectifiable contour in Ω which is holomogous to zero and let $\varphi \in \mathcal{X}'$. Expressing the Riemann integral as a limit of Riemann sums, we see that

$$\varphi\Big(\int_{\mathcal{C}} f(z)\, dz\Big) = \int_{\mathcal{C}} (\varphi \circ f)(z)\, dz.$$

By the scalar-valued version of Cauchy's theorem, the last integral equals zero, as the assumption on f clearly implies that $\varphi \circ f$ is holomorphic. Therefore, by the Hahn-Banach theorem, $\int_{\mathcal{C}} f(z)\, dz = 0$.

PROBLEM 3

(a) We define a function

$$f : \mathbb{C} \to \mathcal{B}(\mathcal{H}), \quad z \mapsto \exp(zN^*)X\exp(-zN^*).$$

It is easy to see that the product of two $\mathcal{B}(\mathcal{H})$-valued analytic functions is analytic, so that f is entire. Since N belongs to the commutant of X, which is a closed subalgebra of $\mathcal{B}(\mathcal{H})$, the operator $\exp(\overline{z}N)$ also commutes with X for each $z \in \mathbb{C}$. Moreover, since $\overline{z}N$ and $-\overline{z}N$ commute, we have

$$\exp(-\overline{z}N)\exp(\overline{z}N) = \exp(0) = 1.$$

From this, we deduce that

$$\begin{aligned}f(z) = \exp(zN^*)X\exp(-zN^*) &= \exp(zN^*)\exp(-\overline{z}N)X\exp(\overline{z}N)\exp(-zN^*) \\ &= \exp(zN^* - \overline{z}N)X\exp(\overline{z}N - zN^*),\end{aligned}$$

where in the last step, we have used that N commutes with N^*. Observe that

$$zN^* - \overline{z}N = 2i\Big(\frac{zN^* - \overline{z}N}{2i}\Big) = 2i\,\mathrm{Im}(zN^*)$$

and that $\mathrm{Im}(zN^*)$ is hermitian. Consequently, $\exp(zN^* - \overline{z}N)$ is unitary, and in particular,

$$\|\exp(zN^* - \overline{z}N)\| \le 1.$$

The same reasoning applies to $\exp(\overline{z}N - zN^*)$, so that

$$\|f(z)\| \le \|\exp(zN^* - \overline{z}N)\|\,\|X\|\,\|\exp(\overline{z}N - zN^*)\| \le \|X\|.$$

This means that f is a bounded entire function. For each $\phi \in \mathcal{X}'$, $\phi \circ f$ is a bounded scalar valued entire function. Thus by Liouville's theorem, each $\phi \circ f$ is constant. By the Hahn-Banach theorem, f is constant. Hence,

$$X = f(0) = f(z) = \exp(zN^*)X\exp(-zN^*)$$

for all $z \in \mathbb{C}$, that is,

$$X\exp(zN^*) = \exp(zN^*)X.$$

Expanding both sides in a power series, we conclude that the coefficients of z must agree (again, by the corresponding scalar-valued fact and the Hahn-Banach theorem), so that

$$XN^* = N^*X.$$

(b) Let $\mathcal{N} = \{N_i, N_i^* : i \in \mathcal{I}\}$. By assumption and by Fuglede's theorem, $\mathcal{N}$ is a self-adjoint set of pairwise commuting operators. Polynomials in these operators thus forms a commutative $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. Hence the norm closure, namely $C^*(\mathcal{N}) = \mathfrak{A}$, is commutative.

For the second part, consider the map

$$\Phi : \mathcal{M}_{\mathfrak{A}} \to \prod_{i\in\mathcal{I}} \sigma(N_i), \quad \varphi \mapsto (\varphi(N_i))_{i\in\mathcal{I}}.$$

It is clear that Φ takes its range in the product of the spectra, and that Φ is continuous. Moreover, since $\mathfrak{A}$ is generated by N_i, and since the characters on $\mathfrak{A}$ are $*$-homomorphisms, each character is uniquely determined by its values on the N_i, so that Φ is injective. Thus, being a continuous injective map from a compact space into a Hausdorff space, Φ is a topological embedding. In particular, $X = \Phi(\mathcal{M}_{\mathfrak{A}})$ is compact. Finally, note that the image of $\pi_i \circ \Phi$ is the set

$$\{\varphi(N_i) : \varphi \in \mathcal{M}_{\mathfrak{A}}\} = \sigma_{\mathfrak{A}}(N_i)$$

by Gelfand theory. Since C*-algebras are inverse closed, the last set equals $\sigma(N_i)$.

(c) If $\mathfrak{A}$ denotes the C*-algebra generated by $U_1, \ldots, U_n$, then by part (b), $\mathfrak{A}$ is commutative and $\mathcal{M}_{\mathfrak{A}}$ can be identified with a compact subset X of $\prod_{i=1}^n \sigma(U_i)$. Since the spectrum of unitaries is contained in the unit circle, we may regard $\mathfrak{A}$ as a compact subset of $\mathbb{T}^n$. Thus, composing the map $\rho : \mathrm{C}(\mathbb{T}^n) \to \mathrm{C}(X)$ given by restriction to X with the inverse of the Gelfand transformation, Γ^{-1} of $\mathrm{C}(X)$ onto $\mathfrak{A}$, we obtain a $*$-homomorphism

$$\Gamma^{-1}\rho : C(\mathbb{T}^n) \to C(\mathcal{M}_{\mathfrak{A}}) \to \mathfrak{A} \subset \mathcal{B}(\mathcal{H}).$$

Observe that the coordinate function z_i restricts to π_i on $\prod_{i=1^n} \sigma(U_i)$, which corresponds to the Gelfand transform $\widehat{U_i}$ by the identification from part (b). Thus, $\rho(z_i) = U_i$ for $i = 1, \ldots, n$, so that

$$\rho(f) = f(U_1, \ldots, U_n)$$

holds for all polynomials $f \in \mathbb{C}[z_1, \ldots, z_n]$, and for arbitrary functions $f \in C(\mathbb{T}^n)$, we may define $f(U_1, \ldots, U_n)$ to be $\rho(f)$. Since *-homomorphisms are contractive, we conclude that

$$\|p(U_1, \ldots, U_n)\| \le \|p\|_{\infty,\mathbb{T}^n}$$

holds for all polynomials p, as asserted.