PM822 Assignment 4 Due Tuesday, March 11

1. (a) Let $1 \in \mathcal{M}$ be an operator space, let $\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ be a completely isometric unital map, and let $\tilde{\varphi}$ be the completely positive extension to the operator system $\mathcal{S} = \overline{\mathcal{M} + \mathcal{M}^*}$. Prove that $\tilde{\varphi}$ is completely isometric.

   (b) Suppose that θ is a unital complete isometry of operator space $\mathcal{S}_1$ onto $\mathcal{S}_2$. Show that a unital cp map $\varphi : \mathcal{S}_2 \to \mathcal{B}(\mathcal{H})$ has the unique extension property if and only if $\varphi\theta : \mathcal{S}_1 \to \mathcal{B}(\mathcal{H})$ has the unique extension property.

2. Let $\mathcal{N} = \big\{\{0\} = N_0 \subset N_1 \subset \cdots \subset N_k = \mathbb{C}^n\big\}$ be a *nest* of subspaces of $\mathbb{C}^n$. Let
$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{M}_n : TN_i \subset N_i \text{ for } 1 \le i \le k\}$$
be the *nest algebra* of $\mathcal{N}$. Let ρ be a contractive unital representation of $\mathcal{T}$. Note that $\mathcal{T}$ contains a copy of $\mathcal{T}_n$ of all upper triangular matrices. Show that a $*$-dilation of $\rho|_{\mathcal{T}_n}$ yields a $*$-dilation of ρ. In particular, ρ is completely contractive.

3. Let $1 \in \mathcal{A}$ be a unital operator algebra contained in $\mathfrak{A} = \mathrm{C}^*(\mathcal{A})$. Let $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely contractive representation. A coextension σ of ρ into $\mathcal{B}(\mathcal{K})$ is *extremal* if every coextension of σ has the form $\tau = \sigma \oplus \tau'$ on $\mathcal{L} = \mathcal{K} \oplus \mathcal{L}'$. Extremal extensions are defined analogously.

   (a) Prove that ρ has an extremal coextension.
   Bonus. If you are careful, you can also arrange that $\mathcal{H}$ is *cyclic*: $\mathcal{K} = \overline{\sigma(\mathcal{A})\mathcal{H}}$.

   (b) Show that a dilation of ρ is a maximal dilation if and only if it is an extremal coextension and an extremal extension.

   (c) Starting with ρ, find an extremal coextension σ_1, then find an extremal extension τ_1 of σ_1. Repeat this process recursively. Show that the limiting process yields a maximal dilation π.

   (d) Show that if ρ is an extremal coextension, and π is a maximal dilation of ρ, then $\mathcal{H}$ is an invariant subspace for $\pi(\mathcal{A})$.
   (*Note: the converse is not true in general. It is sometimes true—see* 5(b).)

4. Let $\{e_n : n \in \mathbb{Z}\}$ be an orthonormal basis for $\mathcal{H}$, and let E_{ij} be the corresponding matrix units. Let $\mathcal{A} = \overline{\mathrm{span}\{I, E_{ii}, E_{2i+1,2i}, E_{2i-1,2i} : i \in \mathbb{Z}\}}$.

   (a) Let $\rho(E_{00}) = \rho(I) = 1$ and $\rho(E_{ij}) = 0$ otherwise be a representation of $\mathcal{A}$ into $\mathbb{C}$. Find all irreducible extremal coextensions and extremal extensions of ρ.

   (b) Show that the process of 3(c) takes countably many steps using cyclic (co)-extensions, but can also be done by one extremal coextension and one extremal extension.

   (c) What is the C*-envelope of $\mathcal{A}$? Find all boundary representations of $\mathcal{A}$.

5. Suppose that an operator algebra $\mathcal{A} \subset \mathrm{C}^*_e(\mathcal{A})$ is *semi-Dirichlet*, meaning $\mathcal{A}^*\mathcal{A} \subset \overline{\mathcal{A} + \mathcal{A}^*}$.

   (a) Let ρ be a representation of $\mathcal{A}$, let π_i be two maximal $*$-dilations of ρ on $\mathcal{L}_i$, let $\mathcal{K}_i = \overline{\pi_i(\mathcal{A})\mathcal{H}}$, and let $\sigma_i(a) = P_{\mathcal{K}_i}\pi(a)|_{\mathcal{K}_i}$. Prove that these two representations are unitarily equivalent via a unitary that fixes $\mathcal{H}$.

   (b) **Bonus.** Use problem 3 to show that the representation σ_1 is extremal.

   (c) Show that if $\mathcal{A}$ and $\mathcal{A}^*$ are both semi-Dirichlet, then $\mathcal{A}$ is Dirichlet (meaning that $\overline{\mathcal{A} + \mathcal{A}^*} = \mathrm{C}^*_e(\mathcal{A})$).