PM822 Assignment 3 Due Tuesday, February 26

1. Let $\mathfrak{A}$ be a non-unital C*-algebra, and let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate identity.
   (a) Let f be a state on $\mathfrak{A}$, and let π_f be the GNS representation. Show that the vectors $(\dot{e}_\lambda)$ form a Cauchy net, and that the limit vector ξ is cyclic for $\pi_f(\mathfrak{A})$.
   (b) Show that if $0 \le a \in \mathfrak{A}$ and $\|a\| = 1$, then there is a state f on $\mathfrak{A}$ such that $f(a) = 1$.
   (c) Deduce (as in the unital case) that $\mathfrak{A}$ has a faithful $*$-representation.

2. (a) Let $\mathfrak{A}$ be a unital C*-algebra, and let $a \in \mathfrak{A}$ with $\|a\| \le 1$. Prove that the map from $\mathbb{C}[z] + \mathbb{C}[\bar{z}]$ into $\mathfrak{A}$ given by $\psi\big(p(z) + q(\bar{z})\big) = p(a) + q(a^*)$ extends to a unital completely positive map of $\mathrm{C}(\mathbb{T})$ into $\mathfrak{A}$.
   (b) Show that if φ is a positive unital map of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$, then $\|\varphi\| = 1$.
   Hint: consider $\varphi \circ \psi$ where ψ comes from (a).

3. Let $\mathfrak{A}, \mathfrak{B}$ be C*-algebras, and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a unital completely positive map. Let (π, V) be the Stinespring dilation of φ
   (a) Show that if $a \in \mathfrak{A}$ and $\varphi(a^*)\varphi(a) = \varphi(a^*a)$, then $V\mathcal{H}$ is invariant for $\pi(a)$. Hence show that $\varphi(ba) = \varphi(b)\varphi(a)$ for all $b \in \mathfrak{A}$.
   (b) Show that $\mathfrak{C} = \{c \in \mathfrak{A} : \varphi(c^*)\varphi(c) = \varphi(c^*c) \text{ and } \varphi(c)\varphi(c^*) = \varphi(cc^*)\}$ is a unital C*-algebra, and $\varphi(c_1 a c_2) = \varphi(c_1)\varphi(a)\varphi(c_2)$ for all $a \in \mathfrak{A}$ and $c_1, c_2 \in \mathfrak{C}$.

4. Let $\mathbb{A} = \{z \in \mathbb{C} : r_1 \le |z| \le r_2\}$ where $0 < r_1 < 1 < r_2 < \infty$. Let $\mathcal{A} = R(\mathbb{A})$. Define a functional on $\mathcal{A}$ by
$$\varphi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$
   (a) Show that there is more than one state on $\mathrm{C}(\partial\mathbb{A})$ extending φ.
   (b) Show that $\mathcal{A}$ is not Dirichlet.
   (c) Show that φ has two inequivalent Stinespring dilations.

5. A von Neumann algebra is a unital C*-subalgebra $\mathfrak{N}$ of $\mathcal{B}(\mathcal{H})$ which is weak-$*$-closed. Say that $\mathfrak{N}$ is *semidiscrete* if there is a net of unital weak-$*$ continuous completely positive maps $\varphi_\lambda : \mathfrak{N} \to \mathcal{M}_{k_\lambda}$ and $\psi_\lambda : M_{k_\lambda} \to \mathfrak{N}$ such that
$$a = \text{w*-}\lim_\lambda \psi_\lambda(\varphi_\lambda(a)) \quad \text{for all} \quad a \in \mathfrak{N}.$$
   Prove that $\mathfrak{N}$ is injective; i.e. if $\varphi : \mathcal{S} \to \mathfrak{N}$ is a completely positive map of an operator system $\mathcal{S}$ into $\mathfrak{N}$ and $\mathcal{T}$ is an operator system containing $\mathcal{S}$, then there is a completely positive map of $\mathcal{T}$ into $\mathfrak{N}$ extending φ.