PM822 Assignment 2 Due Tuesday, February 5

1. Let $T \in \mathcal{B}(\mathcal{H})$ with $\|T\| \le 1$, and let $V \in \mathcal{B}(\mathcal{K})$ be its minimal isometric coextension and let $U \in \mathcal{B}(\mathcal{L})$ be its minimal unitary dilation.

   (a) Show that V is a pure isometry (i.e. no unitary summand) if and only if T^{*n} converges to 0 in the strong operator topology (i.e. $\lim_{n\to\infty} T^{*n}x = 0$ for all $x \in \mathcal{H}$).

   (b) Define $M_+ = (\mathcal{H} \vee U\mathcal{H}) \ominus \mathcal{H}$ and $M_- = (\mathcal{H} \vee U^*\mathcal{H}) \ominus \mathcal{H}$. Show that these subspaces are wandering for U, meaning that $U^n M$ are pairwise orthogonal for all $n \in \mathbb{Z}$. Hence deduce that the restrictions of U to the reducing subspaces $\mathcal{L}_\pm = \bigvee_{n\in\mathbb{Z}} U^n M_\pm$ are unitarily equivalent to the direct sum of copies of the bilateral shift.

   (c) Show that $N := (\mathcal{L}_+ \vee \mathcal{L}_-)^\perp$ is a subspace of $\mathcal{H}$ such that $T|_N$ is unitary.

2. Let $A_1 = A_2 = [0]$ acting on $\mathcal{H} = \mathbb{C}e_0$. Let $\mathcal{K}$ be a Hilbert space containing $\mathcal{H}$. Let V_1 be any pure isometry such that $V_1^* e_0 = 0$. Prove that there is an isometry V_2 commuting with V_1 so that (V_1, V_2) is a minimal isometric coextension of (A_1, A_2).

3. (a) If $T \in \mathcal{B}(\mathcal{H})$ with $\operatorname{spr}(T) < 1$, show that $A = \sum_{n\ge0} T^{*n}T^n$ converges to a positive invertible operator with $A^{-1} \le I$.

   (b) (Rota) Show that $\|A^{1/2}TA^{-1/2}\| \le 1$. (Thus T is similar to a contraction.)

   (c) Show that if K is compact and $\operatorname{spr}(K) \le 1$, then K is similar to $K_0 \oplus T$ where $\operatorname{spr}(K_0) < 1$ and T acts on a finite dimensional space.

   (d) (Sz.Nagy) Show that if K is compact, then K is similar to a contraction if and only if it is power bounded. **Hint:** to deal with T of part (c), use Jordan form.

4. (a) (Douglas) If $\left\|\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]\right\| \le 1$, show that there is a contraction L so that $C = LD_A$. What is the analogue for $\left\|\begin{bmatrix} A & B \end{bmatrix}\right\| \le 1$?

   (b) (Parrott) Suppose that $\left\|\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]\right\| \le 1$ and $\left\|\begin{bmatrix} A & B \end{bmatrix}\right\| \le 1$. Prove that there is a matrix X so that $\left\|\begin{bmatrix} A & B \\ C & X \end{bmatrix}\right\| \le 1$. **Hint:** use $X = -LA^*K$ for certain K, L.

5. (a) Prove that $n \times n$ matrix A_n below has norm

$$\left\|\begin{bmatrix} a_0 & 0 & 0 & \dots & 0 \\ a_1 & a_0 & 0 & \ddots & 0 \\ a_2 & a_1 & a_0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{bmatrix}\right\| \le \inf_{q\in\mathbb{C}[z]} \left\|a_0 + a_1 z + \dots + a_{n-1}z^{n-1} + z^n q(z)\right\|_\infty .$$

   (b) Suppose that $\|A_n\| = 1$. Use 4(b) to find a constant a_n so that the $(n+1)\times(n+1)$ matrix A_{n+1} defined with $a_0, \dots, a_n$ satisfies $\|A_{n+1}\| = 1$.

   (c) (Carathéodory) Hence given $a_0, \dots, a_{n-1}$ with $\|A_n\| = 1$, find a power series

$$h(z) = \sum_{n\ge0} a_n z^n = a_0 + a_1 z + \dots + a_{n-1}z^{n-1} + \text{higher order terms}$$

   so that $\sup_{|z|<1} |h(z)| = 1$. **Hint:** express h as a limit of polynomials which converges uniformly on each disk $\overline{\mathbb{D}}_r = \{z : |z| \le r\}$ for all $r < 1$.