PM822 Assignment 1 Due Tuesday, January 22

1. Let $\mathbb{A}$ denote the set of all continuous functions with absolutely convergent Fourier series with the norm $\|f\|_{\mathbb{A}} := \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$.
   (a) Prove that $\mathbb{A}(\mathbb{T})$ is a Banach algebra which is isometrically isomorphic to $\ell_1(\mathbb{Z})$ with convolution.
   (b) Identify the maximal ideal space of $\mathbb{A}$.
   Hint: for $\varphi \in \mathcal{M}_{\mathbb{A}}$, show that $\lambda = \varphi(z)$ determines φ.
   (c) Prove Wiener's Theorem: if $f \in \mathbb{A}$ and $f(z) \neq 0$ for all $|z| = 1$, then $1/f$ also has an absolutely convergent Fourier series.

2. Let $\mathcal{X}$ be a Banach space. Let Ω be an open subset of $\mathbb{C}$. Let $f : \Omega \to \mathcal{X}$ be a function which is *weakly analytic*, meaning that $\varphi \circ f$ is analytic for every continuous linear functional $\varphi \in \mathcal{X}'$. Fix $z_0 \in \Omega$ and a closed disk $D = \overline{b_r(z_0)} \subset \Omega$.
   (a) Prove that f is bounded on D. **Hint:** uniform boundedness principle.
   (b) Use Cauchy's Theorem to express $\dfrac{\varphi(f(z_0+h)) - \varphi(f(z_0))}{h} - (\varphi\circ f)'(z_0)$ as an integral around the boundary of D for all $|h| < r$. Hence show that for $\|\varphi\| \leq 1$, this difference tends to 0 uniformly as $|h| \to 0$.
   (c) Deduce that $\dfrac{f(z_0+h) - f(z_0)}{h}$ is Cauchy as $h \to 0$, and hence that f is differentiable (i.e. analytic). In particular, f is continuous and therefore Riemann integrable.
   (d) Show that Cauchy's Theorem is valid for $\mathcal{X}$-valued analytic functions: if $\mathcal{C}$ is a rectifiable contour in Ω which is homologous to zero, then $\int_{\mathcal{C}} f(z)\,dz = 0$.
   Remarks: • This is a Riemann integral, which makes sense for continuous functions into $\mathcal{X}$ just as for scalar functions. • All standard results about analytic functions now follow using the classical proofs.

3. (a) (Fuglede) Show that if $N \in \mathcal{B}(\mathcal{H})$ is normal, and commutes with $X \in \mathcal{B}(\mathcal{H})$, then N^* also commutes with X. **Hint:** consider the entire analytic function
   $$f(z) = e^{zN^*}Xe^{-zN^*} = e^{zN^*}e^{-\bar{z}N}Xe^{\bar{z}N}e^{-zN^*}.$$
   (b) Deduce that if $\{N_i : i \in \mathcal{I}\}$ is a family of commuting normal operators, then $\mathfrak{A} = \mathrm{C}^*(\{N_i : i \in \mathcal{I}\})$ is abelian. Show that the maximal ideal space of $\mathfrak{A}$ can be naturally identified with a compact subset X of $\prod_{i\in\mathcal{I}} \sigma(N_i)$ such that the projection $\pi_i(X)$ onto the i-th coordinate is surjective for each $i \in \mathcal{I}$.
   (c) If $U_1, \dots, U_n$ are commuting unitary operators, show that there is a $*$-representation of $\mathrm{C}(\mathbb{T}^n)$ given by $\rho(f) = f(U_1, \dots, U_n)$. Hence deduce that for any polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$, we have $\|p(U_1, \dots, U_n)\| \leq \|p\|_\infty := \sup_{|z_i|=1, 1\leq i\leq n} |p(z_1, \dots, z_n)|$.