PM 352 Assignment 6 Solutions

1. (a) Let $z_1, z_2 \in \Omega$. Then $\gamma(t) = z_1 + (z_2 - z_1)t$ for $0 \le t \le 1$ be the straight line from z_1 to z_2, which lies in Ω because Ω is convex. Observe that

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{1}{z_2 - z_1} \oint_\gamma f'(z)\,dz = \frac{1}{z_2 - z_1} \int_0^1 f'(\gamma(t))\,(z_2 - z_1)\,dt.$$

Therefore

$$\operatorname{Re}\Big(\frac{f(z_2) - f(z_1)}{z_2 - z_1}\Big) = \int_0^1 \operatorname{Re} f'(\gamma(t))\,dt > 0.$$

Hence $f(z_2) \neq f(z_1)$; whence f is one to one.

(b) Let $\log z = \log|z| + i \operatorname{Arg} z$ for $-\pi < \operatorname{Arg} z < \pi$ be the principle branch of the logarithm on $\mathbb{D} \setminus (-1, 0]$. Define $f(z) = z^{1.5} = e^{1.5 \log z}$. Then since $.5 \operatorname{Arg} z \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\operatorname{Re} f'(z) = \operatorname{Re} 1.5 z^{.5} = \operatorname{Re}(1.5|z|^{.5} e^{.5i \operatorname{Arg} z}) = 1.5|z|^{.5} \cos(.5 \operatorname{Arg} z) > 0.$$

However, $f(re^{2\pi i/3}) = -r^{1.5} = f(re^{-2\pi i/3})$ for $0 < r < 1$. Thus f is not one-to-one.

2. (a) $f(z) = e^{iz}$ carries Ω conformally onto the semicircle

$$S = \{z = re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi\}.$$

The Möbius map $Tz = \frac{1+z}{1-z}$ satisfies $T(1) = \infty$, $T(-1) = 0$, $T(0) = 1$ and $T(i) = i$. Hence the real line maps to itself and takes $(-1, 1)$ onto $\mathbb{R}^+$; and the unit circle maps onto the imaginary axis, and the semicircle maps onto $i\mathbb{R}^+$. So T takes S conformally onto the quarter plane $Q = \{z : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$. Finally $g(z) = z^2$ takes Q conformally onto $\mathbb{H}$. So $h(z) = g(T(f(z)))$ is the desired map.

(b) Let $Sz = \frac{z+1}{z-1}$. Then

$$h(z) = \Big(\frac{1 + e^{iz}}{1 - e^{iz}}\Big)^2 = \frac{1 + 2e^{iz} + e^{i2z}}{1 - 2e^{iz} + e^{i2z}} = \frac{e^{-iz} + 2 + e^{iz}}{e^{-iz} - 2 + e^{iz}} = \frac{\cos z + 1}{\cos z - 1} = S(\cos z).$$

Therefore $\cos z = S^{-1}(h(z))$ is a conformal map of Ω onto $S^{-1}\mathbb{H}$. But $S^{-1} = S$ and $S^{-1}(1) = \infty$, $S^{-1}(0) = -1$, $S^{-1}(-1) = 0$, and $S^{-1}(i) = -i$. So S^{-1} takes $\mathbb{R}$ to itself and takes $\mathbb{H}$ onto $-\mathbb{H} = \{z : \operatorname{Im} z < 0\}$.

3. (a) Let $g(z) = f(iz)$ and $h(z) = g^{-1}(if(z))$. Since f takes $\mathbb{D}$ onto S, and multiplication by i carries S conformally onto itself, and g^{-1} maps S conformally onto $\mathbb{D}$, we see that h is a conformal map of $\mathbb{D}$ onto itself. Also $h(0) = g^{-1}(0i) = 0$ and

$$h'(0) = \frac{d}{dz}(g^{-1})(0)\, if'(0) = \frac{if'(0)}{g'(g^{-1}(0))} = \frac{if'(0)}{if'(0)} = 1.$$

By the Schwarz Lemma, $h(z) = z$. Therefore $f(iz) = g(z) = g(h(z)) = if(z)$.

(b)

$$f(z) = \sum_{n=0}^\infty a_n z^n = -if(iz) = (-i)\sum_{n=0}^\infty a_n (iz)^n = \sum_{n=0}^\infty i^{n-1} a_n z^n.$$

By the uniqueness of the power series expansion, $a_n = i^{n-1} a_n$ for all $n \ge 0$. Now $i^{n-1} = 1$ if and only if 4 divides $n - 1$. Therefore $a_n = 0$ except when $n \equiv 1 \pmod 4$.

4. Suppose that a conformal map $f : \Omega \to \mathbb{A}_r$ exists. Then 0 is an isolated singularity, and f is bounded, so it is a removable singularity. Let the extension of f to $\mathbb{D}$ be called f as well. Since $f(\mathbb{D})$ is open, and contained in $\overline{f(\Omega)} = \overline{\mathbb{A}_r}$, $f(0) = w_0$ belongs to $\mathbb{A}_r$. Because f is one-to-one on Ω, $f(z) - w_0$ has a simple zero at 0. Therefore f is one-to-one on $B_\varepsilon(0)$ for some small $\varepsilon > 0$, and $f(\mathcal{B}_\varepsilon(0))$ contains an open ball around w_0. If $f(z_0) = w_0$, then a small ball about z_0 also maps onto a small ball around w_0. So f is not one-to-one. Hence f does not exist.

5. (a) f is analytic on $\mathbb{C} \setminus \{iy : y \le 0\}$, which includes $\overline{\mathbb{H}} \setminus \{0\}$. Also

$$\lim_{|z|\to 0} |f(z)| = \lim_{|z|\to 0} |z|^{1/2}|z-3| = 0 \quad \text{and} \quad \lim_{|z|\to\infty} |f(z)| = \lim_{|z|\to\infty} |z|^{1/2}|z-3| = \infty.$$

(b) $f(x) = \begin{cases} x^{1/2}(x-3) & \text{for } x > 0 \\ i|x|^{1/2}(x-3) & \text{for } x < 0 \end{cases}$. On $x > 0$, $f'(x) = \frac{3}{2}(x^{1/2} - x^{-1/2})$. Therefore f is increasing on $[1,\infty)$ and maps 1–1 onto $[-2,\infty)$, f is decreasing on $[0,1]$, and maps 1–1 onto $[-2,0]$, and similarly, $(-\infty,0]$ is mapped 1–1 onto $-i\mathbb{R}^+ = \{iy : y \le 0\}$.

(c) I claim that f maps $\mathbb{H}$ conformally onto

$$\Omega = \{z : \operatorname{Im} z > 0 \text{ or } \operatorname{Re} z < 0\} \setminus [-2,0).$$

Let $\gamma = [-R,R] \cup \{Re^{it} : 0 \le t \le \pi\}$ for $R > 3$. Then $f \circ \gamma$ is the curve consisting of

$$[-i\sqrt{R}(R+3), 0] \cup [0,-2] \cup [-2, \sqrt{R}(R-3)] \cup \{f(Re^{it}) : 0 \le t \le \pi\}.$$

Moreover $|f(Re^{it})| = \sqrt{R}|Re^{it} - 3| \ge \sqrt{R}(R-3)$ and $\operatorname{Arg} f(Re^{it}) = \frac{t}{2} + \operatorname{Arg}(Re^{it} - 3)$ is increasing from 0 to $3\pi/2$. So this is a large arc from $f(R)$ to $f(-R)$ in Ω. If R is sufficiently large, this curve will encircle any given point $w \in \Omega$ with $\operatorname{Ind}_{f\circ\gamma}(w) = 1$. Thus by the argument principle, the number of points inside γ which are mapped inside $f \circ \gamma$ by f is one. It follows that f maps $\mathbb{H}$ conformally onto Ω.

A (a) Suppose that Ω is a simply connected region in $\mathbb{C}$ containing 0 which has n-fold symmetry in the sense that it is invariant under rotation about 0 through an angle $2\pi/n$. Let f be a conformal map of $\mathbb{D}$ onto Ω with $f(0) = 0$. Then $f(e^{2\pi i/n}z) = e^{2\pi i/n}f(z)$. The argument is again to let $g(z) = f(e^{2\pi i/n}z)$ and $h(z) = g^{-1}(e^{2\pi i/n}f(z)$. Then h is a conformal map of $\mathbb{D}$ onto itself with $h(0) = 0$ and $h'(0) = 1$, whence $h(z) = z$ and $f(e^{2\pi i/n}z) = e^{2\pi i/n}f(z)$. One argues as in 3(b) that if $f = \sum_{k=0}^{\infty} a_k z^k$, then $a_k = 0$ unless $k \equiv 1 \pmod{n}$.

(b) Let f be a conformal map of $\mathbb{D}$ onto Ω with $f(0) = a \in \Omega \cap \mathbb{R}$. Then f takes the circle of radius 1/2 onto a closed curve that has index 1 around $f(0)$. So there is a point on this curve which intersects $[a,\infty)$, say $f(.5e^{i\theta}) = b > a$. Replace f by $g(z) = f(e^{-i\theta}z)$, so that g is a conformal map of $\mathbb{D}$ onto Ω such that $g(0) = a < b = g(1/2)$ are both real.

Let $h(z) = \overline{g(\bar{z})}$. By Assignment 1, problem 1, h is analytic. It is clearly 1–1, and maps $\mathbb{D}$ onto the conjugate of Ω, which is Ω, and $h(0) = a$. Therefore $g^{-1}(h(z))$ is a conformal map of $\mathbb{D}$ onto itself which takes 0 to 0. The only conformal automorphisms of $\mathbb{D}$ taking 0 to itself have the form $e^{i\alpha}z$, so $g^{-1}(h(z)) = e^{i\alpha}z$ for some value of α. Therefore $g(e^{i\alpha}z) = h(z) = \overline{g(\bar{z})}$. Plugging in $z = .5$, we get $g(.5e^{i\alpha}) = \bar{b} = b = g(.5)$. Since g is 1–1, $e^{i\alpha} = 1$. Therefore $g(z) = \overline{g(\bar{z})}$. In particular, $g(z)$ is real if and only if z is real. Thus g maps $(-1,1)$ onto $\Omega \cap \mathbb{R}$.

Alternate: Let f be a conformal map of $\mathbb{D}$ onto Ω with $f(0) = a \in \Omega \cap \mathbb{R}$. Let $h(z) = \overline{g(\bar{z})}$. By Assignment 1, problem 1, h is analytic. It is clearly 1–1, and maps $\mathbb{D}$ onto the conjugate of Ω, which is Ω, and $h(0) = a$. Therefore $f^{-1}(h(z))$ is a conformal map of $\mathbb{D}$ onto itself which takes 0 to 0. The only conformal automorphisms of $\mathbb{D}$ taking 0 to itself have the form $e^{i\alpha}z$, so $f^{-1}(h(z)) = e^{i\alpha}z$ for some value of α. Therefore $f(e^{i\alpha}z) = h(z) = \overline{f(\bar{z})}$. Let $\theta = \alpha/2$ and $g(z) = f(e^{i\theta}z)$. Then g maps $\mathbb{D}$ conformally onto Ω, and

$$g(z) = f(e^{i\theta}z) = f\big(e^{i\alpha}(e^{-i\theta}z)\big) = \overline{f\big(\overline{e^{-i\theta}z}\big)} = \overline{f(e^{i\theta}\bar{z})} = \overline{g(\bar{z})}.$$

In particular, $g(z)$ is real if and only if z is real. Thus g maps $(-1,1)$ onto $\Omega \cap \mathbb{R}$.