PM 352 Assignment 5 Solutions

1. (Type I) Let $\gamma(x) = e^{ix}$ for $0 \le x \le 2\pi$ and set $z = e^{ix}$ and $dz/iz = dx$

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = \frac{1}{4}\int_0^{2\pi} \frac{dx}{a+\sin^2 x} = \frac{1}{4}\oint_\gamma \frac{1}{a+\left(\frac{z^2-1}{2iz}\right)^2}\frac{dz}{iz}$$
$$= i\oint_\gamma \frac{z}{z^4-2(2a+1)z^2+1} = -2\pi \sum_{|w|<1} \operatorname{Res}\left(\frac{z}{z^4-2(2a+1)z^2+1}, w\right)$$

Solve $z^4 - 2(2a+1)z^2 + 1 = 0$ to get $z^2 = 2a+1 \pm 2\sqrt{a+a^2}$. Since $a > 0$, $0 \le 2a+1-2\sqrt{a+a^2}| < 1$. It has two square roots, say $\pm\alpha$. These are simple roots, so

$$\operatorname{Res}\left(\frac{z}{z^4-2(2a+1)z^2+1}, \pm\alpha\right) = \frac{z}{4z^3-4(2a+1)z}\Big|_{z=\pm\alpha}$$
$$= \frac{1}{4\alpha^2-4(2a+1)} = \frac{-1}{8\sqrt{a+a^2}}.$$

Therefore, since there are two equal residues at $\pm\alpha$,

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = (-2\pi)\frac{-2}{8\sqrt{a+a^2}} = \frac{\pi}{2\sqrt{a+a^2}}$$

2. (Type II) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\,dx$. Integrate $\frac{z^2}{(z^2+a^2)^3}$ over the curve $\gamma_R = [-R,R] \cup \{Re^{it} : 0 \le t \le \pi\}$. For large R, the integral over the semicircle is bounded by $2\pi R\frac{R^2}{(R^2+a^2)^3} \le \frac{2\pi}{R^3}$, which goes to 0 as $R \to \infty$. Note that $\frac{z^2}{(z^2+a^2)^3}$ has poles of order 3 at $\pm ai$. Only ai lies in the upper half plane. Observe that setting $Z = z - ai$,

$$\frac{z^2}{(z^2+a^2)^3} = \frac{(Z+ai)^2}{Z^3(Z+2ai)^3} = \frac{Z^2+2aiZ-a^2}{Z^3}\frac{1}{-8a^3i(1+Z/2ai)^3}$$
$$= \frac{(Z^2+2aiZ-a^2)}{Z^3}\frac{(1-Z/2ai-Z^2/4a^2+\dots)^3}{-8a^3i}$$
$$= \frac{(Z^2+2aiZ-a^2)}{Z^3}\frac{(1-3Z/2ai-3Z^2/2a^2+\dots)}{-8a^3i}$$
$$= \frac{-a^2+(2ai+3a/2i)Z-Z^2/2+\dots}{-8a^3iZ^3}$$

Therefore $\operatorname{Res}\left(\frac{z^2}{(z^2+a^2)^3}, ai\right) = \frac{1}{16a^3i}$. So

$$\frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3}\,dx = \pi i \sum_{\operatorname{Im} w>0} \operatorname{Res}\left(\frac{z^2}{(z^2+a^2)^3}, w\right) = \frac{\pi i}{16a^3 i} = \frac{\pi}{16a^3}.$$

3. (Type III) $\displaystyle\int_0^\infty \frac{\cos x}{a^2+x^2}\,dx = \frac{1}{2}\operatorname{Re}\int_{-\infty}^\infty \frac{e^{ix}}{a^2+x^2}\,dx$. Integrate $\frac{e^{iz}}{a^2+z^2}$ over the curve $\gamma_R = [-R,R] \cup \{Re^{it} : 0 \le t \le \pi\}$. A lemma from class shows that the integral over the semicircle goes to 0 as $R \to \infty$. This function has a simple pole at ai in the upper half plane, so $\operatorname{Res}\left(\frac{e^{iz}}{a^2+z^2}, ai\right) = \frac{e^{iz}}{2z}\Big|_{z=ai} = \frac{e^{-a}}{2ai}$. Therefore,

$$\int_0^\infty \frac{\cos x}{a^2+x^2}\,dx = \operatorname{Re}\frac{2\pi i}{2}\frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{2a}.$$

4. (Type IV) Let $I = \int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx$. Consider $f(z) = \frac{z^{1/3}}{1+z^2}$ where we use the branch of $z^{1/3}$ on $\mathbb{C} \setminus \mathbb{R}_+$ given by $(re^{it})^{1/3} = r^{1/3}e^{it/3}$ for $0 < t < 2\pi$. Integrate over the curve

$$\gamma_R = [1/R, R] + \{Re^{it} : 0 \le t \le 2\pi\} - [1/R, R] - \{e^{it}/R : 0 \le t \le 2\pi\}$$

like Fig.1 except for the loop around 1. Observe that $f(z)$ extends from the interior of γ_R to the curve by continuity, and on the return segment $-[1/R, R]$ takes the values $\frac{x^{1/3}e^{2\pi i/3}}{1+x^2}$. Thus in the limit, the integral tends to $-e^{2\pi i/3}I$. Note that $f(z)$ has simple poles at $\pm i$. So $\mathrm{Res}(f, i) = \frac{i^{1/3}}{2i} = \frac{e^{\pi i/6}}{2i}$ and $\mathrm{Res}(f, -i) = \frac{(-i)^{1/3}}{-2i} = \frac{-e^{\pi i/2}}{2i}$.

The integral over $\{Re^{it} : 0 \le t \le 2\pi\}$ is bounded by $2\pi R(R^{1/3})/(R^2 - 1) < 4\pi R^{-2/3}$, which goes to 0 as $R \to \infty$. The integral over $-\{e^{it}/R : 0 \le t \le 2\pi\}$ is bounded by $2\pi/R(R^{-1/3})/(1 - R^{-2}) < 4\pi R^{-2/3}$, which goes to 0 as $R \to \infty$. So in the limit, we obtain

$$(1 - e^{2\pi i/3})I = 2\pi i\Big(\frac{e^{\pi i/6}}{2i} + \frac{-e^{\pi i/2}}{2i}\Big).$$

Simplifying, we get

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = I = \pi\frac{e^{\pi i/6} - e^{\pi i/2}}{1 - e^{2\pi i/3}} = \pi\frac{e^{-\pi i/6} - e^{\pi i/6}}{e^{-\pi i/3} - e^{\pi i/3}} = \pi\frac{\sin \pi/6}{\sin \pi/3} = \frac{\pi}{\sqrt{3}}.$$

5. (Type V) Let $I = \int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$. Using the same curve as problem 4, we integrate the function $f(z) = \frac{\log^2 z}{(1+z^2)^2} = \frac{\log^2 z}{(z-i)^2(z+i)^2}$ using the principal branch of $\log z$ on $\mathbb{C} \setminus \mathbb{R}_+$. This function has double poles at $\pm i$ with residues

$$\begin{aligned}\mathrm{Res}(f, i) &= \frac{d}{dz}\Big(\frac{\log^2 z}{(z+i)^2}\Big)\Big|_{z=i} = \frac{2\log z}{z(z+i)^2} - \frac{2\log^2 z}{(z+i)^3}\Big|_{z=i} \\ &= \frac{\pi i}{-4i} + \frac{2(\pi i/2)^2}{8i} = -\frac{\pi}{4} + \frac{\pi^2 i}{16}\end{aligned}$$

and similarly

$$\mathrm{Res}(f, -i) = \frac{2(3\pi i/2)}{(-i)(-2i)^2} - \frac{2(3\pi i/2)^2}{(-2i)^3} = \frac{3\pi}{4} - \frac{9\pi^2 i}{16}.$$

The integral around the large circle $\{Re^{it} : 0 \le t \le 2\pi\}$ is bounded by

$$2\pi R(\log R + 2\pi)^2/(R^2 - 1) < 8R^{-1}\log^2 R \quad \text{for large } R,$$

and tends to 0 as $R \to \infty$. Similarly, the integral around $-\{e^{it}/R : 0 \le t \le 2\pi\}$ is bounded by $2\pi R^{-1}(\log R + 2\pi)^2/(1 - R^{-2}) < 8R^{-1}\log^2 R$ for large R, and tends to 0 as $R \to \infty$. The integral along $[1/R, R] - [1/R, R]$ yields

$$\int_{1/R}^R \frac{\log^2 x}{(1+x^2)^2}\,dx - \int_{1/R}^R \frac{(\log x + 2\pi i)^2}{(1+x^2)^2}\,dx = \int_{1/R}^R \frac{4\pi^2 - 4\pi i \log x}{(1+x^2)^2}\,dx$$

Thus in the limit,

$$4\pi^2\int_0^\infty \frac{dx}{(1+x^2)^2} - 4\pi i I = 2\pi i\Big(-\frac{\pi}{4} + \frac{\pi^2 i}{16} + \frac{3\pi}{4} - \frac{9\pi^2 i}{16}\Big) = \pi^3 + \pi^2 i.$$

As both integrals are real, we get $\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}$ and $\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$. (Note that the second integral can be done by standard calculus techniques.)

6. Let $I = \displaystyle\int_0^\infty \frac{dx}{1+x^n}$. Integrate $f(z) = \dfrac{1}{1+z^n}$ around Figure 2. Observe that the integral over the arc of the circle is bounded by $\dfrac{2\pi R}{n(R^n-1)}$, which tends to 0 as $R \to \infty$. The integral along the line segment from $Re^{2\pi i/n}$ to 0 is

$$J = -\int_0^R \frac{1}{1+(e^{2\pi i/n}x)^n}\, e^{2\pi i/n}\, dx = -e^{2\pi i/n} I.$$

Now f has a simple pole at $e^{\pi i/n}$ and $\operatorname{Res}(f, e^{\pi i/n}) = \frac{1}{nz^{n-1}}\big|_{z=e^{\pi i/n}} = \frac{-e^{\pi i/n}}{n}$. Therefore $(1-e^{2\pi i/n})I = -\frac{2\pi i e^{\pi i/n}}{n}$. Thus

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{2\pi i}{n(e^{\pi i/n} - e^{-\pi i/n})} = \frac{\pi}{n \sin \pi/n}.$$

Bonus Problems.

A. Let $I = \displaystyle\int_0^\infty \frac{\cos x}{\cosh x}\, dx$. We will integrate $f(z) = \frac{\cos z}{\cosh z} = \frac{e^{iz}+e^{-iz}}{e^z+e^{-z}}$ around the rectangle in Fig.3. On the end segments $\pm R + it$, the integral is bounded by $2\pi \frac{2e^\pi}{e^R - e^{-R}}$, which tends to 0 as $R \to \infty$. Let $I(R) = \displaystyle\int_-^R R^R \frac{\cos x}{\cosh x}\, dx$. The integral $J(R)$ along the segment from $R + \pi i$ to $-R + \pi i$ is

$$\begin{aligned} J(R) &= -\int_{-R}^R \frac{\cos(x+\pi i)}{\cosh(x+\pi i)}\, dx = -\int_{-R}^R \frac{\cos x \cosh \pi - i \sin x \sinh \pi}{\cosh x \cos \pi + i \sinh x \sin \pi}\, dx \\ &= \int_{-R}^R \frac{\cos x \cosh \pi - i \sin x \sinh \pi}{\cosh x}\, dx = \cosh \pi\, I(R) - i \sinh \pi \int_{-R}^R \frac{\sin x}{\cosh x}\, dx \\ &= \cosh \pi\, I(R). \end{aligned}$$

We used the fact that $\frac{\sin x}{\cosh x}$ is an odd function. Now $f(z)$ has simple poles at the zeros of $\cosh z = \cos iz$, namely $z = (2n+1)\pi i/2$ for $n \in \mathbb{Z}$. Only $\pi i/2$ lies inside the rectangle, and

$$\operatorname{Res}(f, \pi i/2) = \frac{\cos z}{-i \sin iz}\Big|_{z=\pi i/2} = \frac{\cosh \pi/2}{i \sin \pi/2} = -i \cosh \pi/2.$$

Hence letting $R \to \infty$,

$$\int_0^\infty \frac{\cos x}{\cosh x}\, dx = \frac{1}{2} \lim_{R\to\infty} I(R) = \frac{1}{2} \frac{(2\pi i)(-i\cosh \pi/2)}{1+\cosh \pi} = \frac{\pi \cosh \pi/2}{1+\cosh \pi} = \frac{\pi/2}{\cosh \pi/2}.$$

B. Integrate $\dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^4}$ around a large square, as shown in class, and obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^4} = -\operatorname{Res}\Big(\frac{\pi \cot \pi z}{(z+\frac{1}{2})^4}, -1/2\Big).$$

Now

$$-\operatorname{Res}\Big(\frac{\pi \cot \pi z}{(z+\frac{1}{2})^4}, -1/2\Big) = -\frac{1}{6}\frac{d^3}{dz^3}\pi \cot \pi z\Big|_{z=-1/2} = \frac{\pi^4(1+2\cos^2 \pi z)}{3\sin^4 \pi z}\Big|_{z=-1/2} = \frac{\pi^4}{3}.$$

Hence

$$\frac{\pi^4}{3} = 2\sum_{n=0}^\infty \frac{1}{(n+\frac{1}{2})^4} = 2\sum_{n=0}^\infty \frac{1}{(n+\frac{1}{2})^4} = 2\sum_{n=0}^\infty \frac{16}{(2n+1)^4}$$

Thus $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$. Now $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}$. So

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \frac{15}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

C. $\displaystyle\int_0^{\infty} \frac{\log x}{x^2-1}\,dx = \frac{\pi^2}{4}$ is obtained by integrating $f(z) = \dfrac{\log^2 z}{z^2-1}$ around Figure 1, picking up half of the residue at 1.

D. (Type III) $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{2x^2}\,dx = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} \frac{1-e^{it}}{t^2}\,dt$, where we substituted $t = 2x$. We will integrate $f(z) = \dfrac{1-e^{iz}}{z^2}$ over the curve

$$\gamma_R = [-R, -1/R] - \{e^{it}R : 0 \le t \le \pi\} + [1/R, R] + \{Re^{it} : 0 \le t \le \pi\}.$$

Observe that the integral of z^{-2} over the large circle is bounded by $\pi R/R^2$ goes to zero; and since $\lim_{|z|\to\infty} \frac{1}{|z|^2} = 0$, a lemma from class shows that the integral of e^{iz}/z^2 also tends to 0 as $R \to \infty$. This function has no poles inside γ_R, and thus the integral is 0. Finally observe that since $1 - e^{0i} = 0$, f has a simple pole at 0 with residue $\frac{d}{dz}(1-e^{iz})\big|_{z=0} = -i$. So another lemma from class shows that the integral over the small semicircle in the positive direction tends to $\pi i \operatorname{Res}(f, 0) = \pi i(-i) = \pi$.

While the function $\frac{\sin^2 x}{x^2}$ extends to be continuous at 0, and thus is integrable, the function $\frac{1-e^{it}}{t^2}$ is not because it has a non-integrable singularity like $1/t$. However, since we are balancing the integral on the positive axis by the negative axis, there is cancelation and the limit exists as a principal value of the integral. (Actually the non-integrable part is pure imaginary, and doesn't affect the real part.) Hence $PV\displaystyle\int_{-\infty}^{\infty} \frac{1-e^{it}}{t^2}\,dt = \pi$, whence $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \pi/2$.

E. $\displaystyle\int_0^{\infty} \frac{\cos x}{a^2-x^2}\,dx = \frac{2(-1)^n}{2n+1}$ for $a = \frac{(2n+1)\pi}{2}$, $n \ge 1$ is obtained by integrating the function $f(z) = \dfrac{e^{iz}}{a^2-z^2}$ over the curve in Figure 4. Note that $\frac{\cos x}{a^2-x^2}$ is continuous on $\mathbb{R}$ and dies off like x^{-2}, and so is integrable. But $f(z)$ has simple poles at $\pm a$. The integral picks up half of the residues at these points.