PM 352 Assignment 4 Solutions

1. Pick a point p in the region R. If $\gamma(t)$ for $0 \le t \le 1$ is a closed curve in R, then the line segments from p to each point on γ_* lies in R. So we can define a homotopy by
$$\Gamma(s,t) = (1-s)\gamma(t) + sp \quad \text{for} \quad 0 \le s,t \le 1.$$
Check that this is continuous and that $\Gamma(s,0) = \Gamma(s,1)$ for $0 \le s \le 1$. This is a homotopy from $\gamma(t) = \Gamma(0,t)$ to the constant curve $p = \Gamma(1,t)$. So R is simply connected.

2. We showed in class that since U be a simply connected, there is a branch $h(z)$ of $\log f(z)$ on U. Let $g(z) = e^{h(z)/2}$. Then $g(z)^2 = e^{h(z)} = f(z)$.

3. Apply Rouché's Theorem. First on the circle of radius 1, using $q(z) = -6z$, we have
$$|p(z) - q(z)| = |z^4 + 3| \le 4 < 6 = |q(z)|.$$
Thus they have the same number of zeros, namely 1. Likewise on the circle of radius 2, using $r(z) = z^4$, we have
$$|p(z) - r(z)| = |-6z + 3| \le 12 + 3 = 15 < 16 = |r(z)|.$$
Again Rouché's Theorem says they have the same number of roots inside the disk, namely 4. Therefore $p(z)$ has 3 roots in the annulus $\mathbb{A}$.

4. (a) $p'(z) = c\displaystyle\sum_{i=1}^{k} n_i(z-a_i)^{n_i-1}\prod_{j\ne i}(z-a_j)^{n_j}$. Thus $\dfrac{p'(z)}{p(z)} = \displaystyle\sum_{i=1}^{k}\frac{n_i}{z-a_i}$.

   (b) Suppose that $p'(b) = 0$. If b is also a root of p, then it clearly lies in $\operatorname{conv} Z(p)$. Ohterwise $0 = \displaystyle\sum_{i=1}^{k}\frac{n_i}{b-a_i} = \sum_{i=1}^{k}\frac{n_i\overline{(b-a_i)}}{|b-a_i|^2}$. Take the conjugate and rearrange to get
$$\sum_{i=1}^{k}\frac{n_i}{|b-a_i|^2}\, b = \sum_{i=1}^{k}\frac{n_i}{|b-a_i|^2}a_i.$$
Let $t_i = \frac{n_i|b-a_i|^{-2}}{\sum_{i=1}^{k} n_i|b-a_i|^{-2}}$. These are positive and $\sum_{i=1}^{k} t_i = 1$. Therefore $b = \sum_{i=1}^{k} t_i a_i$ is a convex combination of the a_i.

5. Note that $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$.

   (i) In $\mathbb{D}$,
$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\frac{1}{1-z/2} + \frac{1}{1-z} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}(1 - 2^{-n-1})z^n. \end{aligned}$$

   (ii) In $\mathbb{A}$,
$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\frac{1}{1-z/2} - \frac{1}{z}\frac{1}{1-1/z} \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} -2^{-n-1}z^n + \sum_{n=-\infty}^{-1} z^n. \end{aligned}$$

(iii) In U,

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{z}\frac{1}{1-2/z} - \frac{1}{z}\frac{1}{1-1/z}$$
$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty}(2^n-1)z^{-n-1} = \sum_{n=-\infty}^{-1}(2^{-k-1}-1)z^n.$$

6. (a) $\tan z = \dfrac{\sin z}{\cos z}$ and $0 = \cos z = \dfrac{e^{iz}-e^{-iz}}{2}$ exactly when $e^{2iz} = -1$, namely when $2iz = (2n+1)\pi i$ or $z = (2n+1)\frac{\pi}{2}$. These are the poles of $\tan z$. Since $\frac{d}{dz}\cos z = -\sin z$ and $\sin(2n+1)\frac{\pi}{2} = (-1)^n \neq 0$, it follows that $\cos z$ has a simple zero at these points. Thus these are poles of order one.

(b) Since $\sin z$ is entire, $z\sin(1/z)$ is analytic on $\mathbb{C}\setminus\{0\}$. Expanding about $z=0$, we get

$$z\sin(1/z) = z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-2n-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{-2n}.$$

Since there are infinitely many non-zero terms, this is an essential singularity.

(c) Here $f(z)$ is analytic on $\mathbb{H}\setminus\{1\}$. We know from class that $\log z$ has a Taylor expansion about $z=1$ that converges on $B_1(1)$, and has the form $\log z = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n}(z-1)^n$. Therefore,

$$f(z) = \sum_{n=-2}^{\infty}\frac{(-1)^n}{n+3}(z-1)^n = (z-1)^{-2} - \frac{1}{2}(z-1)^{-1} + \dots.$$

This is a pole of order 2.

A. Suppose that f is not constant, but $f(p)=0$ for some $p \in U$. Then this is an isolated zero. Thus there is a small $r>0$ so that $f(z)\neq 0$ on $\overline{B_r(p)}\setminus\{p\}$. In particular, $|f(z)| \geq \delta > 0$ on the circle $\gamma(t) = p + re^{it}$, $0 \leq t \leq 2\pi$, by the Extreme Value Theorem.

Since f_n converge u.c.c. to f, there is an N so that $\sup\{|f(z)-f_n(z)| : |z-p| \leq r\} < \delta/2$ for all $n \geq N$. So $|f(z)-f_n(z)| < |f(z)|$ on γ. By Rouché's Theorem, f and f_n have the same number of zeros in $B_r(p)$. However f has one zero and by hypothesis f_n has none. This contradiction shows that f has no zeros unless it is the zero function.
(*Note: this can occur by taking* $f_n = 1/n$ *all constant functions.*)

B. Wlog, the two points are ± 1. Let $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ be the union of two circles $\mathcal{C}_\pm = \sigma_{\pm *} = \{z : |z \mp 1| = 1\}$ where $\sigma_\pm(t) = \pm 1 \mp e^{it}$ for $0 \leq t \leq 2\pi$. One can show that every curve γ is homotopic to a curve in $\mathcal{C}$. The given curve is seen to be homotopic to $\sigma_+ + \sigma_- - \sigma_+ - \sigma_-$. Look at $\{t : \gamma(t) \neq 0\}$. This splits the interval into countably many open segments. Since γ is uniformly continuous, all but finitely many are too short to wrap around one of the two circles—so these segments can be homotoped to 0. This shows that γ is homotopic to a finite sum of the four curves $\pm\sigma_\pm$. Now you need to argue that except for permuting these words in a cycle, one cannot change the order of the words unless one has consecutive terms σ_+ and $-\sigma_+$ or σ_- and $-\sigma_-$, which can be collapsed to 0.