PM 352 Assignment 3 Solutions

1. (a) Parallel lines intersect only at the point ∞. So under a Möbius map with $T\infty = 0$ and $T\mathbb{R}_\infty = \mathbb{R}_\infty$, we must get a family of circles that touch $\mathbb{R}$ only at 0. So this is the family of circles which are tangent to $\mathbb{R}$ at 0. You can also see that this forces the centre to lie on $i\mathbb{R}$. So the circles can alternatively be describes as $C_t = \{z = x + iy : x^2 + (y - t)^2 = t^2\}$ for $t \in \mathbb{R} \setminus \{0\}$ together with $\mathbb{R}$.

   (b) This family of lines pass through ∞ and cross $\mathbb{R}$ orthogonally. So the image under T is the family of circles through 0 crossing $\mathbb{R}$ orthogonally. These can be described as the circles tangent to the imaginary axis at 0. As in (a), this can be described as the family of circles with centre $x \in \mathbb{R}$ and radius $|x|$, for $x \in \mathbb{R} \setminus \{0\}$ together with $i\mathbb{R}$.

2. (a) If $Tz = e^{i\theta} \frac{z-a}{1-\bar{a}z}$, then because complex conjugates have the same modulus,

$$|T(e^{is})| = |e^{i\theta}| \left| \frac{e^{is} - a}{1 - \bar{a}e^{is}} \right| = \left| \frac{e^{is} - a}{e^{-is} - \bar{a}} \right| = 1.$$

   So T takes the unit circle onto itself. Since $|T0| = |a| < 1$, it follows that the disk inside this circle is mapped to itself.

   Now suppose that S is a Möbius map taking $\mathbb{D}$ onto itself. Say $S0 = a \in \mathbb{D}$. Compose S with the map T above so that $TS(0) = 0$. Then if $TSz = \frac{az+b}{cz+d}$, we find that $b = 0$ and WLOG, $a = 1$. Since $1 = |TS(e^{it})| = 1/|ce^{it} + d|$, we see that $|ce^{it} + d| = 1$ for all $t \in \mathbb{R}$. This is the locus of a circle centred at d. So either $d = 0$ and $|c| = 1$, or $c = 0$ and $|d| = 1$. In the first case, TS is constant, a contradiction. So $c = 0$ and $TSz = \bar{d}z$. Hence $S^{-1} = dT$ has the desired form. Hence so does S. (**Alternatively,** you can construct T as above and apply Schwarz's Lemma to TS and $S^{-1}T^{-1}$ to see that $TSz = e^{i\theta}z$.)

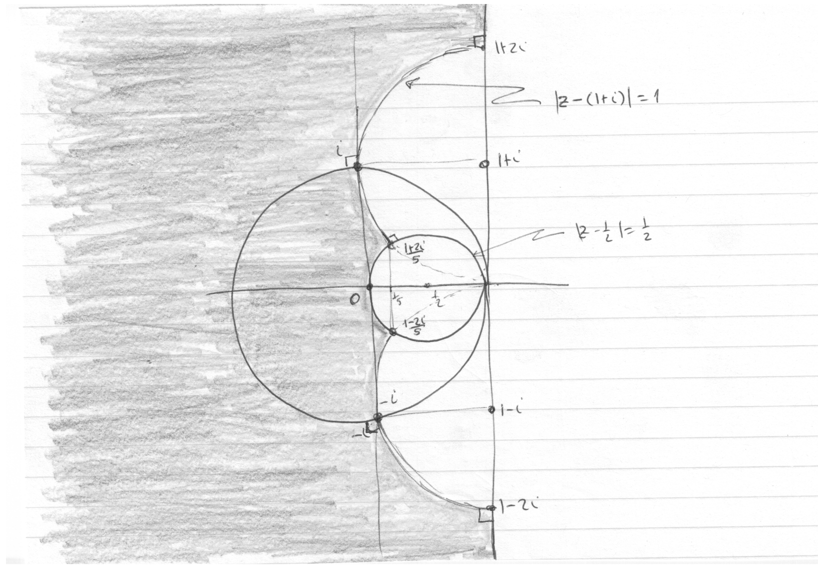
   (b) Translate and dilate $\mathcal{C}_1$ until it is mapped to the unit circle. If $\mathcal{C}_2$ is outside, perform an inversion $z \to 1/z$ to get it inside. The rotate until its centre a lies on $(-1, 1)$. Let the radius of this new circle $\mathcal{C}_2'$ be r, $0 < r < 1$. If $a = 0$, we are done. Otherwise apply a Möbius map $Tz = \frac{z-b}{1-bz}$ for $b \in (-1, 1)$. This maps the unit circle to itself and $\mathbb{R}$ to itself. As $\mathcal{C}_2'$ intersects $\mathbb{R}$ orthogonally, so does $T\mathcal{C}_2'$; and hence the centre lies on $(-1, 1)$. The two intersection points are $T(a \pm r) = \frac{a-b\pm r}{1-ab\mp br}$. Solving for $T(a + r) = -T(a - r)$ yields $b^2 - \left(\frac{a^2+1-r^2}{a}\right)b + 1 = 0$. This has two roots with product 1, so pick the one inside the disk. This is the desired map.

3. (a) Observe that $T\infty = 1$, $T(-1) = \infty$, $T0 = -1$ and $T1 = 0$. The Möbius map $Tz = \frac{z-1}{z+1}$ has the same effect on these points. But the image of three points determines a Möbius map. So this is it.

   (b) The edges of the square are mapped to arcs of circles that meet each other orthogonally. The two vertical segments are lines orthogonal to $\mathbb{R}$. The line through -1 goes through ∞, and thus is the line $1 + i\mathbb{R}$. Since $T(-1 \pm i) = 1 \pm 2i$, the left segment is mapped to $\{1 + iy : |y| \geq 2\}$. The segment $[1-i, 1+i]$ is sent to part of the circle $\mathcal{C}$ centred on $\mathbb{R}$ through 1 and 0. The line segments $[-1+i, 1+i]$ and $[-1-i, 1-i]$ are sent to circles tangent to $\mathbb{R}$ at 1. They passing through i and $1 + 2i$, and $-i$ and $1 - 2i$ respectively. So these are the circles $\mathcal{C}_\pm = \{x + iy : x^2 + (y \mp 1)^2 = 1\}$. Calculate $\mathcal{C} \cap \mathcal{C}_+ = \{1, \frac{1+2i}{5}\}$ and $\mathcal{C} \cap \mathcal{C}_- = \{1, \frac{1-2i}{5}\}$. The line segment $[-1+i, 1+i]$ is sent to the arc of $\mathcal{C}_+$ from $\frac{1+2i}{5}$ through i to $1 + 2i$. Similarly, the line segment $[-1 + i, 1 + i]$ is sent to the arc of $\mathcal{C}_-$ from $\frac{1+2i}{5}$ through $-i$ to $1 - 2i$. The image of the square includes $T(-1, 1) = (-\infty, 0)$. So we see that T takes the square to the set

$$\{x + iy : x < 1,\ (x - \tfrac{1}{2})^2 + y^2 > \tfrac{1}{4},\ x^2 + (y - 1)^2 > 1 \text{ and } x^2 + (y + 1)^2 > 1\}.$$

   See the figure on the next page.



4. We can argue as we did for integrating $1/z$ around a circle that $\oint_\gamma \frac{dz}{z} = 2\pi i$. On the other hand,

$$\begin{aligned} 2\pi i &= \oint_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{\gamma'(t)\,dt}{\gamma(t)} \\ &= \int_0^{2\pi} \frac{(-a\sin t + ib\cos t)}{(a\cos t + ib\sin t)}\frac{(a\cos t - ib\sin t)}{(a\cos t - ib\sin t)}\,dt = \int_0^{2\pi} \frac{(b^2-a^2)\sin t\cos t + 2abi}{a^2\cos^2 t + b^2\sin^2 t}\,dt \\ &= (b^2-a^2)\int_0^{2\pi} \frac{\sin t\cos t}{a^2\cos^2 t + b^2\sin^2 t}\,dt + 2abi\int_0^{2\pi}\frac{dt}{a^2\cos^2 t + b^2\sin^2 t} \end{aligned}$$

Equating the imaginary part of both sides yields the desired equality.

5. $f(z)$ has a power series expansion around a of the form $f(z) = \sum_{k\geq 0} \frac{f^{(k)}(a)}{k!}(z-a)^k$ valid on a disk of positive radius ρ around a (but possibly not valid on all of $\overline{B_r(p)}$. Nevertheless, we can write $f(z) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(z-a)^k + (z-a)^{n+1}g(z)$ where $g(z) = \big(f(z) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(z-a)^k\big)/(z-a)^{n+1}$ is clearly analytic on $\Omega\setminus\{a\}$, and also on $B_\rho(a)$ where it has a power series expansion. Therefore

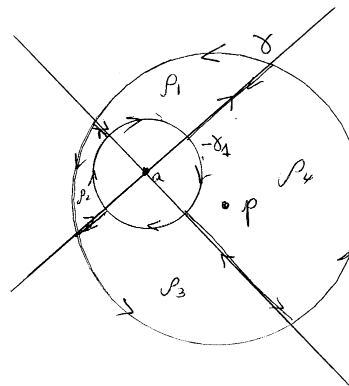
$$\begin{aligned} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz &= \oint_\gamma \frac{\sum_{k=0}^n f^{(k)}(a)(z-a)^k/k! + (z-a)^{n+1}g(z)}{(z-a)^{n+1}}\,dz \\ &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}\oint_\gamma \frac{dz}{(z-a)^{n+1-k}} + \oint_\gamma g(z)\,dz \\ &= \sum_{k=0}^{n-1} 0 + 2\pi i\frac{f^{(n)}(a)}{n!} + 0 = \frac{2\pi i}{n!}f^{(n)}(a). \end{aligned}$$

The first n integrals are 0 because $(z-a)^{-n-1+k}$ has an analytic primitive in a region around γ, namely $(z-a)^{-n+k}/(k-n)$; and the last integral is 0 by Cauchy's Integral Theorem.

Alternate solution. We showed in class that if $\gamma_1(t) = a + se^{it}$ for $0 \leq t \leq 2\pi$ is a small circle around a (say $s < r - |a-p|$), then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{\gamma_1} \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

To get the desired integral, it suffices to show that $\oint_{\gamma-\gamma_1} \frac{f(z)}{(z-a)^{n+1}}\,dz = 0$. This can be accomplished by drawing lines through a at 45 degree angles to the axes to split each circle into four arcs. Connect the end points by line segments. Then $\gamma - \gamma_1$ is split into the sum of four *closed* curves $\sum_{i=1}^{4} \rho_i$, and each lies in a nice convex region on which $\frac{f(z)}{(z-a)^{n+1}}$ is analytic. Thus these four integrals are all zero by Cauchy's Integral Theorem.



6. Apply Cauchy's estimates for the derivatives using the disk of radius R for large R. We obtain

$$|f^{(k)}(0)| \le \lim_{R\to\infty} \frac{k!(A+BR^n)}{R^k} = 0 \quad \text{for} \quad k > n.$$

Therefore the power series about 0 is a polynomial of degree at most n. Since f is entire, this series expansion is valid on all of $\mathbb{C}$. So f is a polynomial.

A. This is known as the *reflection principle.*

Define $\tilde{f}(z) = f(z)$ on U, $\tilde{f}(z) = \overline{f(\overline{z})}$ for $z \in V$, and $f(x) = \lim_{z\in U,\, z\to x} f(z)$ for $x \in (a,b)$. By hypothesis, $\tilde{f}(z)$ is analytic on U and continuous on $U \cup (a,b)$. By Assignment 1, #1, $\tilde{f}(z)$ is analytic on V ; and also it is easy to see that $\tilde{f}(z)$ is continuous on $V \cup (a,b)$. So $\tilde{f}(z)$ is continuous on $U \cup (a,b) \cup V$.

To see that $\tilde{f}(z)$ is analytic, we use Morera's Theorem. Consider a closed rectangle R contained in $U \cup (a,b) \cup V$ with boundary curve γ. If this lies entirely within U or V, it integrates to 0 by Cauchy's Integral Theorem. So suppose that it cuts the real axis. Split it into two rectangles using the real axis as one boundary. The integral over γ is the sum of the integral around these two rectangles. So it suffices to suppose that one boundary edge lies on (a,b), say $[c,d]$ where $a < c < d < b$, and the rectangle lies in $U \cup (a,b)$.

Let $\varepsilon > 0$. Let $M = \max\{|\tilde{f}(z)| : z \in R\}$. Use the uniform continuity of $\tilde{f}$ on $S = [c,d]\times[0,r] \subset U \cup (a,b)$ to find $0 < \delta < \varepsilon$ so that $w, z \in S$ and $|w - z| \le \delta$ implies that $|\tilde{f}(w) - \tilde{f}(z)| < \varepsilon$. Split the rectangle into two using $x + \delta i$ as an edge, say $R = R_1 \cup R_2$, where R_1 lies in U and $R_2 = [c,d] \times [0,\delta]$. Let γ_i be the boundary curves. Then $\oint_{\gamma_1} \tilde{f}(z)\,dz = 0$ by Cauchy's Integral Theorem. Finally,

$$\begin{aligned}
\left| \oint_{\gamma_1} \tilde{f}(z)\,dz \right| &= \left| \int_c^d \tilde{f}(x) - \tilde{f}(x+\delta i)\,dx + i\int_0^\delta \tilde{f}(d+iy) - \tilde{f}(c+iy)\,dy \right| \\
&\le \int_c^d |\tilde{f}(x) - \tilde{f}(x+\delta i)|\,dx + \int_0^\delta |\tilde{f}(d+iy)| + |\tilde{f}(c+iy)|\,dy \\
&\le (d-c)\varepsilon + \delta(2M) \le (2M + d - c)\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\oint_\gamma \tilde{f}(z)\,dz = 0$. Hence $\tilde{f}$ is analytic ny Morera's Theorem.