

PM 352 Assignment 2 Solutions

1. If $a = 0$, there are no solutions. If $a = re^{i\theta} \neq 0$, then $\frac{1}{z} = \log r + i(\theta + 2k\pi)$ for some $k \in \mathbb{Z}$. Hence $z = \frac{1}{\log r + i(\theta + 2k\pi)}$ for $k \in \mathbb{Z}$. Thus this set of solutions contains complex numbers of arbitrarily small modulus, so there are solutions in \mathbb{A}_r for any $r > 0$. It follows that $f(\mathbb{A}_r) = \mathbb{C} \setminus \{0\}$.

2. (a) Start with the right hand side

$$\begin{aligned} \sin z \cos w + \cos z \sin w &= \frac{(e^{iz} - e^{-iz})}{2i} \frac{(e^{iw} + e^{-iw})}{2} + \frac{(e^{iz} + e^{-iz})}{2} \frac{(e^{iw} - e^{-iw})}{2i} \\ &= \frac{e^{i(z+w)} + e^{i(z-w)} - e^{i(-z+w)} + e^{i(-z-w)} + e^{i(z+w)} - e^{i(z-w)} + e^{i(-z+w)} + e^{i(-z-w)}}{4i} \\ &= \frac{e^{i(z+w)} + e^{i(-z-w)}}{2i} = \sin(z+w). \end{aligned}$$

(b) $\sin(iy) = \frac{e^{-y} - e^y}{2i} = i \frac{e^y - e^{-y}}{2} = i \sinh y$ and $\cos(iy) = \frac{e^{-y} + e^y}{2} = \cosh y$. So

$$\begin{aligned} |\sin(x+iy)|^2 &= |\sin x \cos(iy) + \cos x \sin(iy)|^2 = |\sin x \cosh y + i \cos x \sinh y|^2 \\ &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \sinh^2 y. \end{aligned}$$

The case of $|\cos(x+iy)|^2$ is similar.

(c) If $2 = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$, let $w = e^{iz}$ and rearrange this identity to be $w^2 - 4iw - 1 = 0$. This quadratic has two roots $w = 2i \pm \sqrt{-3} = (2 \pm \sqrt{3})i$. Now solve $e^{iz} = (2 \pm \sqrt{3})i$ to get $iz = \log(2 \pm \sqrt{3}) + i(\frac{\pi}{4} + 2k\pi)$ for $k \in \mathbb{Z}$. So $z = \frac{\pi}{4} + 2k\pi - i \log(2 \pm \sqrt{3})$ for $k \in \mathbb{Z}$.

3. If $f = u + iv$, then $h := \log |f| = \log \sqrt{u^2 + v^2} = \frac{1}{2} \log(u^2 + v^2)$. Compute

$$h_x = \frac{uu_x + vv_x}{u^2 + v^2} \quad \text{and} \quad h_y = \frac{uu_y + vv_y}{u^2 + v^2}.$$

$$h_{xx} = \frac{(u^2 + v^2)(uu_{xx} + u_x^2 + vv_{xx} + v_x^2) - 2(uu_x + vv_x)^2}{(u^2 + v^2)^2}$$

and

$$h_{yy} = \frac{(u^2 + v^2)(uu_{yy} + u_y^2 + vv_{yy} + v_y^2) - 2(uu_y + vv_y)^2}{(u^2 + v^2)^2}$$

Using the Cauchy-Riemann equations $v_x = -u_y$ and $v_y = u_x$, we get

$$\begin{aligned} \Delta h &= h_{xx} + h_{yy} \\ &= \frac{(u^2 + v^2)(u\Delta u + v\Delta v) + 2(u^2 + v^2)(u_x^2 + u_y^2) - 2(uu_x - vv_y)^2 - 2(uu_y + vv_x)^2}{(u^2 + v^2)^2} \\ &= \frac{2(u^2 + v^2)(u_x^2 + u_y^2) - 2(u^2u_x^2 - 2uvu_xu_y + v^2u_y^2 + u^2u_y^2 + 2uvu_xu_y + v^2u_x^2)}{(u^2 + v^2)^2} \\ &= \frac{2(u^2 + v^2)(u_x^2 + u_y^2) - 2(u^2u_x^2 + v^2u_y^2 + u^2u_y^2 + v^2u_x^2)}{(u^2 + v^2)^2} = 0. \end{aligned}$$

Hence $\log |f|$ is harmonic.

4. (a) Suppose that $g(z)$ is analytic on U such that $g'(z) = f(z)$. Then $\frac{d}{dt}g(e^{it}) = g'(e^{it})ie^{it} = if(e^{it})e^{it}$. Therefore by the Fundamental Theorem of Calculus,

$$\int_0^{2\pi} f(e^{it})e^{it} dt = -i \int_0^{2\pi} \frac{d}{dt}g(e^{it}) dt = -ig(e^{it}) \Big|_0^{2\pi} = -i(g(1) - g(1)) = 0.$$

(b) $\int_0^{2\pi} \frac{1}{e^{it}} e^{it} dt = \int_0^{2\pi} 1 dt = 2\pi \neq 0$. By (a), there is no primitive for $f(z)$ defined on \mathbb{A} .

5. The geometric series $f(z) = \sum_{n \geq 0} (z+1)^n = \frac{1}{1-(z+1)} = \frac{1}{z}$ provided $|z+1| < 1$; i.e. the radius of convergence is 1. Using term by term differentiation, we obtain

$$\frac{d}{dz} \left(\frac{1}{z} \right) = \frac{-1}{z^2} = \sum_{n \geq 1} n(z+1)^{n-1} = \sum_{n \geq 0} (n+1)(z+1)^n.$$

This is also valid on $B_1(-1)$. Differentiating a second time yields

$$\frac{d}{dz} \left(\frac{-1}{z^2} \right) = \frac{2}{z^3} \sum_{n \geq 1} n(n+1)(z+1)^{n-1}.$$

Hence $g(z) = (z+1) \sum_{n \geq 1} n(n+1)(z+1)^{n-1} = \frac{2(z+1)}{z^3}$. This is valid in $\{z : |z+1| < 1\}$.

6. Let $z_1 = z_0 + Re^{i\theta}$. To say that the power series converges absolutely at z means that

$$\sum_{n=0}^{\infty} |a_n| |z_1 - z_0|^n = \sum_{n=0}^{\infty} |a_n| R^n < \infty.$$

Now $u_n(z) = a_n(z - z_0)^n$ satisfies

$$\|u_n\|_{\overline{B_R(z_0)}} = \sup\{|u_n(z)| : |z - z_0| \leq R\} = |a_n| R^n.$$

Since this sequence is summable, the Weierstrass M-test shows that the series

$$\sum_{n \geq 0} a_n(z - z_0)^n = \sum_{n \geq 0} u_n(z)$$

converges uniformly on $\overline{B_R(z_0)} = \{z : |z - z_0| \leq R\}$.

7. Without loss of generality, $z_0 = 0$. Observe that for $z \neq w$ in $B_r(0)$ with $0 < r < R$,

$$\frac{f(z) - f(w)}{z - w} = a_1 + \sum_{n \geq 2} a_n \frac{z^n - w^n}{z - w}.$$

Note that

$$\left| \frac{z^n - w^n}{z - w} \right| = |z^{n-1} + z^{n-2}w + \dots + zw^{n-2} + w^{n-1}| < nr^{n-1}.$$

Hence

$$\left| \frac{f(z) - f(w)}{z - w} \right| > |a_1| - \sum_{n \geq 2} n|a_n|r^{n-1}.$$

Now the radius of convergence of $\sum_{n \geq 2} n|a_n|z^{n-1}$ is R , and thus $g(r) = \sum_{n \geq 2} n|a_n|r^{n-1}$ is a continuous function on $[0, R)$ with $g(0) = 0$. Pick $r_1 \in (0, R)$ so that $g(r_1) < |a_1|/2$. Then for $z, w \in B_{r_1}(0)$, we have

$$\left| \frac{f(z) - f(w)}{z - w} \right| > |a_1| - g(r_1) > \frac{|a_1|}{2}.$$

In particular, $f(z) \neq f(w)$.

A. (a) By Hadamard's Theorem, $1/R_1 = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ and $1/R_2 = \limsup_{n \rightarrow \infty} |b_n|^{1/n}$. For any $r < R$, there is an integer N so that

$$|a_n|^{1/n} < 1/r \quad \text{and} \quad |b_n|^{1/n} < 1/r \quad \text{for all } n \geq N.$$

Equivalently, $|a_n| < r^{-n}$ and $|b_n| < r^{-n}$ for $n \geq N$. Let $C = \max\{1, |a_n|r^n, |b_n|r^n : 0 \leq n < N\}$. Then $|a_n| \leq Cr^{-n}$ and $|b_n| \leq Cr^{-n}$ for all $n \geq 0$. Hence

$$|c_n| \leq \sum_{k=0}^n |a_k b_{n-k}| \leq \sum_{k=0}^n (Cr^{-k})(Cr^{k-n}) = (n+1)C^2 r^{-n}.$$

Therefore

$$\limsup_{n \rightarrow \infty} |c_n|^{1/n} \leq \limsup_{n \rightarrow \infty} ((n+1)C^2 r^{-n})^{1/n} = \frac{1}{r}.$$

By Hadamard's Theorem, the radius of convergence is at least r . Since $r < R$ is arbitrary, the radius of convergence is at least R .

(b) If $|z - z_0| = r < R$, then the series for $f(z)$ and $g(z)$ both converge absolutely. Thus

$$\sum_{k=0}^{\infty} |a_k| r^k \sum_{l=0}^{\infty} |b_l| r^l < \infty.$$

Therefore the double series $\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (a_k z^k)(b_l z^l)$ converges absolutely. It follows that all rearrangements converge to the same sum. Therefore

$$\begin{aligned} f(z)g(z) &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (a_k z^k)(b_l z^l) = \sum_{n=0}^{\infty} \sum_{k=0}^n (a_k z^k)(b_{n-k} z^{n-k}) \\ &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^n a_k b_{n-k} = \sum_{n=0}^{\infty} c_n z^n = h(z). \end{aligned}$$

(c) Consider $f(z) = \sum_{n \geq 0} z^n$ and $g(z) = 1 - z$. Observe that f has radius of convergence 1, and sums to $f(z) = \frac{1}{1-z}$. Hence $h(z) = f(z)g(z) = 1$, which has radius of convergence infinity.