# PM 352 Assignment 1 Solutions

1. Write $f(x+iy) = u(x,y) + iv(x,y)$. Then

$$g(x+iy) = \overline{f(x-iy)} = u(x,-y) - iv(x,-y) =: \tilde{u}(x,y) + i\tilde{v}(x,y).$$

Thus $\tilde{u}(x,y) = u(x,-y)$ and $\tilde{v}(x,y) = -v(x,-y)$. Therefore

$$\tilde{u}_x = \frac{\partial}{\partial x}(u(x,-y)) = u_x(x,-y), \qquad \tilde{u}_y = \frac{\partial}{\partial y}(u(x,-y)) = -u_y(x,-y),$$
$$\tilde{v}_x(x,y) = \frac{\partial}{\partial x}(-v(x,-y)) = -v_x(x,-y)\ , \text{ and } \ \tilde{v}_y(x,y) = \frac{\partial}{\partial y}(-v(x,-y)) = v_y(x,-y).$$

By the Cauchy-Riemann equations for $f$, we have $u_x = v_y$ and $u_y = -v_x$. So

$$\tilde{u}_x = u_x(x,-y) = v_y(x,-y) = \tilde{v}_y(x,y) \text{ and}$$
$$\tilde{u}_y = -u_y(x,-y) = v_x(x,-y) = -\tilde{v}_x(x,y).$$

Thus $g$ satisfies the Cauchy-Riemann equations, and thus is analytic.

2. Since $g(i) = -i = e^{3\pi i/2} = e^{9\pi i/6}$, we see that we must consider $-1$ as having argument $9\pi$. Tracing around, we find that the argument of $i$ is $13\pi/2$ while the argument of $3i/4$ is $9/pi/2$. Thus $g(-1) = \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{3}+1}{2\sqrt{2}} - i\frac{\sqrt{3}-1}{2\sqrt{2}}$ and
$g(3i/4) = \left(\frac{3}{4}\right)^{1/6}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \left(\frac{3}{4}\right)^{1/6}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -\frac{3^{1/6}}{2^{5/6}} + \frac{3^{1/6}}{2^{5/6}}i$

3. (a) $\frac{\partial}{\partial z}z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy) = \frac{1}{2}(1+1) = 1$. Similarly, $\frac{\partial}{\partial \bar{z}}z = 0$, $\frac{\partial}{\partial z}\bar{z} = 0$ and $\frac{\partial}{\partial \bar{z}}\bar{z} = 1$.
By the product rule, $\frac{\partial}{\partial \bar{z}}(z^j\bar{z}^k) = jz^{j-1}\frac{\partial z}{\partial z}\bar{z}^k + z^jk\bar{z}^{k-1}\frac{\partial \bar{z}}{\partial z} = jz^{j-1}\bar{z}^k$.
Similarly, $\frac{\partial}{\partial \bar{z}}(z^j\bar{z}^k) = kz^j\bar{z}^{k-1}$.

(b) This is evident for $m = n = 0$. Assume that the formula is known for all $j,k$ and $(m-1,n)$. Then

$$\frac{\partial^m}{\partial z^m}\frac{\partial^n}{\partial \bar{z}^n}(z^j\bar{z}^k) = \frac{\partial}{\partial z}\Big(\frac{\partial^{m-1}}{\partial z^m}\frac{\partial^n}{\partial \bar{z}^n}(z^j\bar{z}^k)\Big) = \frac{\partial}{\partial z}\Big(\frac{j!k!}{(j+1-m)!(k-n)!}z^{j+1-m}\bar{z}^{k-n}\Big)$$
$$= \frac{j!k!}{(j+1-m)!(k-n)!}(j+1-m)z^{j-m}\bar{z}^{k-n} = \frac{j!k!}{(j-m)!(k-n)!}z^{j-m}\bar{z}^{k-n}.$$

Obtaining the $(m,n)$ case from the $(m,n-1)$ case is similar. So by induction, the formula is valid for all $(m,n)$.

(c) A polynomial vanishes everywhere only if $a_{jk} = 0$ for all $j,k \geq 0$. Now $\frac{\partial p}{\partial \bar{z}} = \sum_{j=0}^N \sum_{k=1}^N ka_{jk}z^j\bar{z}^{k-1}$. So $\frac{\partial p}{\partial \bar{z}} \equiv 0$ if and only if $a_{jk} = 0$ for all $j$ and all $k \geq 1$.

4. (a) Compute $\varphi'(z) = \dfrac{-1(1+z) - 1(1-z)}{(1+z)^2} = \dfrac{-2}{(1+z)^2}$. Thus $\varphi$ is differentiable on $\mathbb{C}\setminus\{-1\}$ and the derivative is never 0.

(b) If $w = \frac{1-z}{1+z}$, then $w + wz = 1 - z$; so $z = \frac{1-w}{1+w} = \varphi(w)$. Thus $\varphi(\varphi(z)) = \varphi(w) = z$. It follows that $\varphi$ is one-to one because it is determined by $w$. Also every $z$ is in the range of $\varphi$. So $\varphi$ maps $\mathbb{C}\setminus\{-1\}$ one-to-one and onto itself.

(c) Let $z = r\cos\theta + ir\sin\theta$. Compute

$$\varphi(z) = \frac{(1 - r\cos\theta) - ir\sin\theta}{(1 + r\cos\theta) + ir\sin\theta} = \frac{(1 - r\cos\theta) - ir\sin\theta}{(1 + r\cos\theta) + ir\sin\theta}\,\frac{(1 + r\cos\theta) - ir\sin\theta}{(1 + r\cos\theta) - ir\sin\theta}$$
$$= \frac{(1 - r^2\cos^2\theta - r^2\sin^2\theta) + i(2r\cos\theta\sin\theta)}{(1 + r\cos\theta)^2 + r^2\sin^2\theta} = \frac{(1 - r^2) + ir\sin 2\theta}{(1 + r\cos\theta)^2 + r^2\sin^2\theta}.$$

Since the denominator is positive, this lies in the right half plane if and only if $r < 1$. By (b), the whole right half plane is in the range of $\varphi$. So the unit disk must be mapped onto $\mathbb{H}$.

5. (a) Solve $z + \frac{1}{z} = w$ or $z^2 - wz + 1 = 0$ to get $z = \dfrac{w \pm \sqrt{w^2 - 4}}{2}$. Except for $w^2 = 4$ or $w = \pm 2$ (where $z = \pm 1$ are double roots), this has two solutions $z_1$ and $z_2$; and $z_2 = 1/z_1$ by inspection. So the image of the circles of radius $r$ and $1/r$ coincide. Thus the restriction of $f$ to $\mathbb{A}$ is two-to-one except at the points $\pm 1$. Since $f'(z) = 1 - z^{-2}$, these points are the zeros of $f'$.

   (b) $f(r\cos\theta + ir\sin\theta) = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$ lies on the curve $\frac{x^2}{(r+1/r)^2} + \frac{y^2}{(r-1/r)^2} = 1$, which is an ellipse. Then $c^2 = (r + \frac{1}{r})^2 - (r - \frac{1}{r})^2 = 4$. So the foci are $\pm 2$. So an alternative description of the ellipse is $\{z : |z-2| + |z+2| = 2(r + 1/r)\}$. When $r = 1$, we get $f(\cos\theta + i\sin\theta) = 2\cos\theta$, which maps onto $[-2, 2]$ (and in fact is 2-to-1 except at the endpoints).

   (c) The images of the circles of radius $1 < r < 2$ are distinct ellipses with the same foci. They coincide with the image of the circles of radius $1/r$. The circle of radius 1 maps onto the line segment $[-2, 2]$. Since $r + 1/r$ is monotone increasing on $r \geq 1$, and takes $[1, 2)$ onto $[4, 5)$, the image ellipses fill in the solid ellipse

$$f(\mathbb{A}) = \{z : |z-2| + |z+2| < 5\} = \{z = x + iy : \frac{x^2}{(5/2)^2} + \frac{y^2}{(3/2)^2} < 1\}.$$