PM 352 **Assignment 6** **Due Wednesday March 30.**

1. (a) Let Ω be a *convex* region. Suppose that $f(z)$ is analytic on Ω and $\operatorname{Re} f'(z) > 0$ for all $z \in \Omega$. Prove that f is one to one.
   Hint: find an expression for $\frac{f(z_2)-f(z_1)}{z_2-z_1}$ as an line integral.

   (b) Show by example that (a) may fail for non-convex regions.
   Hint: consider a power of z on $\Omega = \mathbb{D} \setminus (-1, 0]$.

2. Let $\Omega = \{z : 0 < \operatorname{Re} z < \pi,\ \operatorname{Im} z > 0\}$.

   (a) Find an explicit conformal map of Ω onto $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$.
   Hint: use a combination of an exponential, a Möbius map and a power.

   (b) Express your map in terms of the cosine function. Hence show that $\cos z$ maps Ω conformally onto $-\mathbb{H}$.

3. Let $S = \{z : |\operatorname{Re} z| < 1 \text{ and } |\operatorname{Im} z| < 1\}$ be the unit square. Let f be a conformal map of $\mathbb{D} = \{z : |z| < 1\}$ onto S such that $f(0) = 0$.

   (a) Prove that $f(iz) = if(z)$. i.e. f has four-fold symmetry.
   Hint: Let $g(z) = f(iz)$ and $h(z) = g^{-1}(if(z))$. Apply Schwarz's Lemma to h.

   (b) Let the power series expansion of f be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Prove that $a_n = 0$ when $n \not\equiv 1 \pmod 4$.

4. Show that there is no conformal map of the region $\Omega = \{z : 0 < |z| < 1\}$ onto any annulus $\mathbb{A}_r = \{z : 1 < |z| < r\}$ for $1 < r < \infty$.
   Hint: such a map would have an isolated singularity at 0. Remember that f is open. Where is $f(0)$? What maps there?

5. Let $z^{1/2}$ be the principle branch of the square root in $\mathbb{C} \setminus \{iy : y \le 0\}$, the plane slit along the negative imaginary axis. Define $f(z) = z^{1/2}(z - 3)$.

   (a) Show that f is analytic on $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$, continuous on $\overline{\mathbb{H}}$ and $\lim\limits_{|z|\to\infty} |f(z)| = \infty$.

   (b) Find the image of the real axis.

   (c) Find $\Omega = f(\mathbb{H})$, and show that f maps $\mathbb{H}$ conformally onto Ω.

A. **Bonus Problem. Please hand in separately.**

   (a) Generalize question 3 as much as you can.

   (b) Suppose that Ω is a simply connected region which is symmetric about the the real line $\mathbb{R}$ (i.e. $z \in \Omega$ implies $\bar{z} \in \Omega$). Prove that there is a conformal map of $\mathbb{D}$ onto Ω which takes $(-1, 1)$ onto $\Omega \cap \mathbb{R}$.