PM 352 **Assignment 4** **Due Wednesday March 10.**

1. Show that every *convex* region is simply connected.

2. Let U be a simply connected open set in $\mathbb{C}$, and suppose that $f(z)$ is analytic on U and never vanishes. Show that there is an analytic function $g(z)$ on U such that $g(z)^2 = f(z)$.

3. How many zeros does $p(z) = z^4 - 6z + 3$ have in the annulus $\mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

4. Let $p(z) = c\prod_{i=1}^{k}(z - a_i)^{n_i}$ be a non-constant polynomial.
   (a) Obtain a simplified formula for $\frac{p'(z)}{p(z)}$.
   (b) Suppose that $p'(b) = 0$. Use (a) to express b as a *convex combination* of the zeros $a_1, \dots, a_k$ of p. i.e. $Z(p') \subseteq \overline{\mathrm{conv}(Z(p))}$. **Hint:** use the formula for $0 = \frac{p'(b)}{p(b)}$ from (a), rationalize each term so that the denominator is positive, and then solve for b.

5. Find the Laurent expansion for $f(z) = \dfrac{1}{(z-1)(z-2)}$ is the regions (i) $\mathbb{D} = \{z : |z| < 1\}$, (ii) $\mathbb{A} = \{z : 1 < |z| < 2\}$ and (iii) $U = \{z : |z| > 2\}$.

6. Classify the singularities, including orders of poles, for these functions:
   (a) $\tan z$ (b) $z \sin(1/z)$,
   (c) $f(z) = \dfrac{\log z}{(z-1)^3}$ where $\log z$ is the principle branch of the logarithm defined in $\mathbb{H} = \{z : \mathrm{Re}\, z > 0\}$ (so that $\log 1 = 0$).

Bonus Problems. Please hand in separately.

A. Let $f_n(z)$ be analytic functions on a connected open set U. Suppose that f_n never vanishes for $n \geq 1$, and that they converge u.c.c. to an (analytic) function $f(z)$. Prove that either $f = 0$ or f has no zeros at all.
Hint: count the zeros of f_n in a small disk around an isolated zero of f.

B. The curve in the figure is homologous to zero in the plane with the two points removed. Show that it is not homotopic to a point.