PM 352 **Assignment 3** **Due Friday February 12.**

1. Suppose that T is a Möbius map which takes $\mathbb{R}$ to itself and sends ∞ to 0.
   (a) What is the image of the family of lines parallel to $\mathbb{R}$?
   (b) What is the image of the family of lines perpendicular to $\mathbb{R}$?

2. (a) Show that a Möbius map takes $\mathbb{D} = \{z : |z| < 1\}$ onto itself if and only if it has the form $Tz = e^{i\theta} \frac{z-a}{1-\bar{a}z}$ for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.
   (b) Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two disjoint circles in $\mathbb{C}$. Show that there is a Möbius map T so that $T\mathcal{C}_1$ and $T\mathcal{C}_2$ are concentric.
   Hint: first map $\mathcal{C}_1$ onto the unit circle so that $T\mathcal{C}_2$ is inside the unit disk with its centre on the real axis. Then use a Möbius map from (a) which also maps $\mathbb{R}$ to $\mathbb{R}$. Remember that the centre of a circle is not preserved by these maps, so you need another way to determine whether the circles are concentric.

3. (a) If the Riemann sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ is rotated one quarter turn about the y-axis so that the north pole goes to $(1, 0, 0)$, what is the corresponding Möbius map T on $\mathbb{C}$ induced by the sterographic projection of this motion of the sphere?
   (b) Let Q be the square with vertices $\pm 1 \pm i$. Find the image of Q under the map T.

4. Let a, b be positive real numbers. Let $\gamma(t) = a \cos t + ib \sin t$ for $0 \le t \le 2\pi$ be an ellipse. Evaluate $\displaystyle\oint_\gamma \frac{dz}{z}$ in two ways, and deduce that $\displaystyle\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}$.

5. Suppose that $f(z)$ is analytic in an open set containing the closed ball $\overline{B_r(p)}$. Let γ denote the boundary circle of $\overline{B_r(p)}$ oriented anticlockwise. Show that for $a \in B_r(p)$ that
$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz.$$
   Hint: use the power series expansion to split $\frac{f(z)}{(z-a)^{n+1}}$ into the first few terms plus an analytic piece.

6. Let $f(z)$ be an entire function. Suppose that there are real constants A and B so that $|f(z)| \le A + B|z|^n$ for all $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most n.

A. **Bonus Problem. Please hand in separately.**
Let U be an open subset of the upper half plane such that $\overline{U} \cap \mathbb{R} = [a, b]$. Suppose that f is analytic on U and extends to be a continuous function on $U \cup (a, b)$ taking *real* values on (a, b). Prove that there is an analytic function $\tilde{f}$ defined on $U \cup (a, b) \cup V$, where $V = \{\bar{z} : z \in U\}$, such that $\tilde{f}|_U = f$.