

1. Let $f(z) = e^{1/z}$ for $z \neq 0$. Let $r > 0$, and set $\mathbb{A}_r = \{z \in \mathbb{C} : 0 < |z| < r\}$. Determine the range $f(\mathbb{A}_r)$. **Hint:** Solve $e^{1/z} = a$.
 2. Define $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ for $z \in \mathbb{C}$.
 - (a) Show that $\sin(w + z) = \sin z \cos w + \cos z \sin w$ for all $w, z \in \mathbb{C}$.
 - (b) Notice that $\sin(iy) = i \sinh y$ and $\cos(iy) = \cosh y$ for $y \in \mathbb{R}$. Show that for $x, y \in \mathbb{R}$,

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y \quad \text{and} \quad |\cos(x + iy)|^2 = \cos^2 x + \sinh^2 y.$$
 - (c) Solve $\sin z = 2$.
 3. Prove that if f is analytic, then the real-valued function $\log |f(z)|$ is harmonic. **Hint:** write $f(x + iy) = u(x, y) + iv(x, y)$.
 4. (a) Let f be analytic on an open set U containing the unit circle $\mathbb{T} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Show that if f has an analytic primitive defined on all of U , then $\int_0^{2\pi} f(e^{it})e^{it} dt = 0$.
 (b) Let $f(z) = 1/z$ on $\mathbb{A} = \{z : \frac{1}{2} < |z| < 2\}$. Compute $\int_0^{2\pi} f(e^{it})e^{it} dt$. Use this to show that there is no branch of the logarithm defined on \mathbb{A} .
 5. Let $g(z) = \sum_{n \geq 1} (n^2 + n)(z + 1)^n$. Find the radius of convergence and a closed form formula for $g(z)$. Justify your steps.
 6. Let $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$ be a power series which has a finite radius of convergence $R > 0$. Suppose that there is some z_1 with $|z_1 - z_0| = R$ such that the series converges absolutely at z_1 . Prove that the series converges uniformly on the closed disk $\overline{B_R(z_0)}$.
 7. Let $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$ be a power series which has a positive radius of convergence R . Prove that if $a_1 \neq 0$, then f is one-to-one on a small ball, $B_r(z_0)$, around z_0 . **Hint:** WLOG, $z_0 = 0$. Look at $\frac{f(z) - f(w)}{z - w}$.
- A. **Bonus Problem. Please hand in separately.**
 You are given two power series, $f(z) = \sum_{n \geq 0} a_n z^n$, which has radius of convergence $R_1 > 0$, and $g(z) = \sum_{n \geq 0} b_n z^n$, which has radius of convergence $R_2 > 0$. Define a new power series by setting $c_n = \sum_{k=0}^n a_k b_{n-k}$ for $n \geq 0$, and $h(z) = \sum_{n \geq 0} c_n z^n$.
- (a) Show that $h(z)$ has radius of convergence *at least* $R = \min\{R_1, R_2\}$.
 - (b) Prove that $h(z) = f(z)g(z)$ for $|z| < \min\{R_1, R_2\}$.
 - (c) Find an example for which the radius of convergence of the series for h is greater than R .