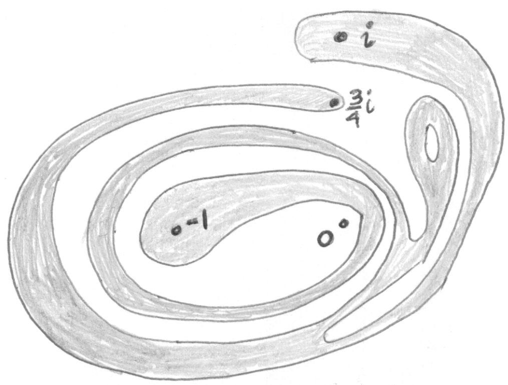
PM 352 **Assignment 1** **Due Monday January 18.**

1. Let U be a non-empty open subset of $\mathbb{C}$, and let f be an analytic function on U. Define g on $\tilde{U} = \{z : \bar{z} \in U\}$ by $g(z) = \overline{f(\bar{z})}$. Prove that g is analytic.
Hint: check the Cauchy-Riemann equations.

2. Let U be the region below. Suppose that g is an analytic function on U such that $g(z)^6 = z$ and $g(-1) = -i$. Find $g(i)$ and $g(3i/4)$ in Cartesian form.



3. (a) Compute $\frac{\partial}{\partial z} z$, $\frac{\partial}{\partial \bar{z}} z$, $\frac{\partial}{\partial z} \bar{z}$ and $\frac{\partial}{\partial \bar{z}} \bar{z}$. Hence find a formula for $\frac{\partial}{\partial z}(z^j \bar{z}^k)$ and $\frac{\partial}{\partial \bar{z}}(z^j \bar{z}^k)$.
   (b) Show by induction that for $m, n \geq 0$,
$$\frac{\partial^m}{\partial z^m} \frac{\partial^n}{\partial \bar{z}^n}(z^j \bar{z}^k) = \begin{cases} \frac{j!k!}{(j-m)!(k-n)!} z^{j-m} \bar{z}^{k-n} & \text{if } m \leq j \text{ and } n \leq k \\ 0 & \text{if } m > j \text{ or } n > k. \end{cases}$$
   (c) Let $p(z, \bar{z}) = \sum_{j,k=0}^N a_{jk} z^j \bar{z}^k$ be a polynomial in z and $\bar{z}$. Show that $\frac{\partial p}{\partial \bar{z}} \equiv 0$ (i.e. p is analytic) if and only if $p = \sum_{j=0}^N a_{j0} z^j$ (i.e. no $\bar{z}$ terms at all).

4. Define $\varphi(z) = \dfrac{1-z}{1+z}$ for $z \neq -1$.
   (a) Show that φ is analytic on $\mathbb{C} \setminus \{-1\}$ and that $\varphi'(z)$ is never 0.
   (b) Solve $\varphi(z) = w$. Deduce that $\varphi(\varphi(z)) = z$, and that φ maps $\mathbb{C} \setminus \{-1\}$ one-to-one and onto itself.
   (c) Show that φ maps the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ one-to-one and onto the right half plane $\mathbb{H} = \{z : \operatorname{Re} z > 0\}$. **Hint:** take z in polar form and compute $\operatorname{Re} \varphi(z)$.

5. Let $\mathbb{A} = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$, and let $f(z) = z + \frac{1}{z}$.
   (a) Show that f is 2-to-1 except at the zeroes of f'.
   (b) Show that the image of the circle of radius r is an ellipse except for $r = 1$. What is the image in this case? **Hint:** An ellipse is the locus of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or, setting $c^2 = a^2 - b^2$, the set $\{z = x + iy \in \mathbb{C} : |z - c| + |z + c| = a\}$.
   (c) Show that $f(A)$ is the interior of an ellipse.