Math 249, Winter 2013
Assignment 8

Not to be handed in.

1. Let G be a a graph, and let M be a matching.

   (a) Let M' be another matching, and let H be the spanning subgraph of G with $E(H) = M \cup M'$. Prove that every component of H is one of the following: (i) an isolated vertex; (ii) a single edge that lies in $M \cap M'$; (iii) an even cycle with edges alternately in M and M'; or (iv) a path with edges alternately in M and M'.

   (b) If M is not a maximum matching, prove that an augmenting path exists.

2. The $a \times b$ Knight's Move graph, denoted $KM(a,b)$ has vertex set $[a] \times [b]$ and $\{(s,t),(u,v)\}$ is an edge iff $(s-u)^2 + (t-v)^2 = 5$. Note that $KM(a,b)$ is a subgraph of $KM(a',b')$ if $a \leq a'$ and $b \leq b'$.

   (a) Use the Bipartite Matching Algorithm to find a perfect matching in the following graphs:

      (i) $KM(3,4)$

      (ii) $KM(3,6)$

      (iii) $KM(5,6)$

   For (i), start with a perfect matching of $KM(2,4)$. For (ii) and (iii), use the previous solution as a starting point.

   (b) Prove that $KM(a,2b)$ has a perfect matching, for $a \geq 3$, $b \geq 2$.

   [*Hint:* Hall's Theorem probably won't help. Use your solutions from (a) as the building blocks for a solution to (b).]

3. Find a 3-regular graph with no perfect matching. Prove that your answer is correct.

   [*Hint:* If a graph has a bridge, then that bridge must either be in every perfect matching, or not be in any perfect matching. Why? If a graph has multiple bridges in awkward places, this can prevent the graph from having perfect matching.]

4. Let G be a graph, and let A be its adjacency matrix. If $\det(A)$ is odd, prove that G has a perfect matching.

   [*Hint:* This is quite hard, but not impossible. You'll need the Leibniz formula for $\det(A)$:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}\,.$$

   Here A is an $n \times n$ matrix, S_n is the set of all permutations of $[n]$, and $\operatorname{sgn}(\sigma) = \pm 1$ is the sign of σ. Some of the permutations in S_n can be thought of as perfect matchings in K_n. Which ones correspond to matchings in G? How does this relate to the Leibniz formula? Once you understand the answers to these questions, let $m(G)$ be the number of perfect matchings of G. Prove that $\det(A) \equiv m(G) \pmod 2$.]