Math 249, Winter 2013
Assignment 6

Due Wednesday, March 20, in class.

1. Let $p \in \mathbb{N}$, and let $n_1, n_2, n_3, \dots, n_p$ be a sequence of non-negative integers such that

$$\sum_{i=1}^{p} n_i = p \qquad \text{and} \qquad \sum_{i=1}^{p} i n_i = 2p - 2\,.$$

   Prove that there exists a tree with p vertices that has n_i vertices of degree i for all $i \in [p]$.

   (In other words the obvious restrictions on the degrees of the vertices in a tree are the only restrictions.)

2. Let G be a graph with vertex set $V(G) = [p]$. Let A be the adjacency matrix of G. For all $k \geq 0$, and $i, j \in [p]$, prove (by induction) that $(A^k)_{ij}$ is equal to the number of walks of length k in G from vertex i to vertex j.

3. Let V be the set of binary strings of length 5 with an even number of ones. Let H be the graph with vertex set $V(H) = V$, where $\sigma, \tau \in V$ are adjacent if σ and τ differ in all but one place. (For example 01100 and 00011 are adjacent in H, since the strings differ in all but the first place.) Let B be the 16×16 adjacency matrix of H,

   (a) A graph G is *arc-transitive* if for any two pairs of adjacent vertices (u, v) and (x, y) there is an automorphism f of G such that $f(u) = x$ and $f(v) = y$. Prove that both H and its complement are arc-transitive.

   (b) Prove that $B^2 + 2B = 3I + 2J$, where I and J are the 16×16 identity matrix and all ones matrix respectively.

   (c) Find the eigenvalues of B, and determine their multiplicities.

   (d) Suppose J is a 3-regular subgraph of H. Prove that one of the following must be true: either J is connected, or J has two components, both of which are induced subgraphs.

   [*Hint:* Let C be the adjacency matrix of J. If J is spanning but not connected, one can show using spectral theory that there must be a non-zero vector x such that $Cx = 3x$ and $Bx = x$. Don't worry about trying to prove this, but see if you can complete the argument from here.]