Math 249, Winter 2013
Assignment 4

Due Wednesday, February 27, in class.

1. Let $\mathcal{S}$ be a set of combinatorial objects and let $w_1 : \mathcal{S} \to \mathbb{N}$ and $w_2 : \mathcal{S} \to \mathbb{N}$ be weight functions. Recall that the multivariate generating series for $\mathcal{S}$ with respect to w_1 and w_2 is defined to be
$$\Phi_{\mathcal{S}}(x,y) = \sum_{\sigma \in \mathcal{S}} x^{w_1(\sigma)} y^{w_2(\sigma)} .$$
Prove the following:
   (a) The number of objects $\sigma \in \mathcal{S}$ such that $w_1(\sigma) = n$ and $w_2(\sigma) = m$ is $[x^n y^m]\Phi_{\mathcal{S}}(x,y)$.
   (b) $[y^m]\Phi_{\mathcal{S}}(x,y)$ is the generating series for $\{\sigma \in \mathcal{S} \mid w_2(\sigma) = m\}$ with respect to w_1.
   (c) If $w_3 : \mathcal{S} \to \mathbb{N}$ is a weight function and there exist constants k_1, k_2 such that $w_3(\sigma) = k_1 w_1(\sigma) + k_2 w_2(\sigma)$ for all $\sigma \in \mathcal{S}$, then $\Phi_{\mathcal{S}}(t^{k_1}, t^{k_2})$ is the generating series for $\mathcal{S}$ with respect to w_3.
   (d) (Product Lemma) Suppose $\mathcal{S} = \mathcal{A} \times \mathcal{B}$, where $\mathcal{A}$ is a set with weight functions α_1, α_2, and $\mathcal{B}$ is a set with weight functions β_1, β_2. Suppose there exist constants $\gamma_1, \gamma_2 \in \mathbb{Z}$ such that
$$w_1(a,b) = \alpha_1(a) + \beta_1(b) + \gamma_1 \qquad \text{and} \qquad w_2(a,b) = \alpha_2(a) + \beta_2(b) + \gamma_2$$
for all $(a,b) \in \mathcal{S}$. Then
$$\Phi_{\mathcal{S}}(x,y) = x^{\gamma_1} y^{\gamma_2} \Phi_{\mathcal{A}}(x,y) \Phi_{\mathcal{B}}(x,y) .$$

2. For $k \geq 0$, let $\mathcal{D}_k \subset \{0, 1, \ldots, k\}^*$ be the set of strings whose blocks all have odd length. Define $k+1$ weight functions $w_0, w_1, \ldots, w_k$ on $\mathcal{D}_k$, where $w_i(\sigma)$ is the number of i's in σ.
Let $g(u) = u(1 + u - u^2)^{-1}$. Prove the following.
   (a) $\Phi_{\mathcal{D}_0}(x) = \dfrac{1}{1 - g(x)}$
   (b) $\Phi_{\mathcal{D}_1}(x,y) = \dfrac{1}{1 - g(x) - g(y)}$
   (c) $\Phi_{\mathcal{D}_2}(x,y,z) = \dfrac{1}{1 - g(x) - g(y) - g(z)}$

3. Prove that the number of 01-strings of length n that do not have 00100 as a substring is
$$[x^n] \frac{1 + x^3 + x^4}{1 - 2x + x^3 - x^4 - x^5} .$$

4. Let $\mathcal{R}$ be the set of 012-strings that do not have any of the following as a substring:

$$012 \quad 021 \quad 102 \quad 120 \quad 201 \quad 210\,.$$

In other words, every substring of length three must have a repeated digit. For example, $001121220 \in \mathcal{R}$, $22011 \notin \mathcal{R}$. Let $\mathcal{R}_n \subset \mathcal{R}$ denote the set of strings in $\mathcal{R}$ of length n. Note that $\mathcal{R}_2$ is the set of all 012-strings of length 2.

Define a 9×9 matrix A whose rows and columns are indexed by strings $\alpha = \alpha_1\alpha_2$ and $\beta = \beta_1\beta_2$ in $\mathcal{R}_2$:

$$A = \begin{bmatrix} A_{00,00} & A_{00,01} & A_{00,02} & A_{00,10} & A_{00,11} & A_{00,12} & A_{00,20} & A_{00,21} & A_{00,22} \\ A_{01,00} & A_{01,01} & A_{01,02} & A_{01,10} & A_{01,11} & A_{01,12} & A_{01,20} & A_{01,21} & A_{01,22} \\ A_{02,00} & A_{02,01} & A_{02,02} & A_{02,10} & A_{02,11} & A_{02,12} & A_{02,20} & A_{02,21} & A_{02,22} \\ A_{10,00} & A_{10,01} & A_{10,02} & A_{10,10} & A_{10,11} & A_{10,12} & A_{10,20} & A_{10,21} & A_{10,22} \\ A_{11,00} & A_{11,01} & A_{11,02} & A_{11,10} & A_{11,11} & A_{11,12} & A_{11,20} & A_{11,21} & A_{11,22} \\ A_{12,00} & A_{12,01} & A_{12,02} & A_{12,10} & A_{12,11} & A_{12,12} & A_{12,20} & A_{12,21} & A_{12,22} \\ A_{20,00} & A_{20,01} & A_{20,02} & A_{20,10} & A_{20,11} & A_{20,12} & A_{20,20} & A_{20,21} & A_{20,22} \\ A_{21,00} & A_{21,01} & A_{21,02} & A_{21,10} & A_{21,11} & A_{21,12} & A_{21,20} & A_{21,21} & A_{21,22} \\ A_{22,00} & A_{22,01} & A_{22,02} & A_{22,10} & A_{22,11} & A_{22,12} & A_{22,20} & A_{22,21} & A_{22,22} \end{bmatrix}$$

where

$$A_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha_2 = \beta_1 \text{ and } \alpha_1\beta_1\beta_2 \in \mathcal{R}_3 \\ 0 & \text{otherwise.} \end{cases}$$

Let I denote the 9×9 identity matrix, and let $\mathbf{1}$ denote the all ones column vector of size 9.

(a) Explain why $|\mathcal{R}_3|$ is sum of the entries in the matrix A.

(b) For $\beta \in \mathcal{R}_2$, $k \geq 0$, let $\mathcal{R}_{k+2,\beta}$ be the set of strings in $\mathcal{R}_{k+2}$ that end in β. Show that

$$|\mathcal{R}_{2,\beta}| = 1\,, \qquad |\mathcal{R}_{k+3,\beta}| = \sum_{\alpha \in \mathcal{R}_2} |\mathcal{R}_{k+2,\alpha}| A_{\alpha,\beta}\,.$$

(c) For $k \geq 0$, write

$$\mathbf{1}^t A^k = \begin{bmatrix} N_{00}^k & N_{01}^k & N_{02}^k & N_{10}^k & N_{11}^k & N_{12}^k & N_{20}^k & N_{21}^k & N_{22}^k \end{bmatrix}.$$

For $\beta \in \mathcal{R}_2$, show that

$$N_\beta^0 = 1\,, \qquad N_\beta^{k+1} = \sum_{\alpha \in \mathcal{R}_2} N_\alpha^k A_{\alpha,\beta}\,.$$

(d) Deduce that $|\mathcal{R}_{k+2,\beta}| = N_\beta^k$, and $|\mathcal{R}_{k+2}| = \mathbf{1}^t A^k \mathbf{1}$, for $k \geq 0$.

(e) Show that

$$\Phi_{\mathcal{R}}(x) = 1 + 3x + x^2\left(\mathbf{1}^t(I - xA)^{-1}\mathbf{1}\right).$$

[*Hint:* The geometric series formula is valid here!]

(f) Try to simplify this approach: find a 2×2 matrix that can replace A (you may need to modify a few other things too). Hence show that

$$\Phi_{\mathcal{R}}(x) = 1 + 3x + x^2\left(\frac{9 + 3x}{1 - 2x - x^2}\right).$$

Do not give a detailed proof, but briefly explain your reasoning.