Math 249, Winter 2013
Assignment 3

Due Wednesday, February 13, in class.

1. Let $\mathcal{S}$ be a set of combinatorial objects with a weight function. Prove that the number of elements of $\mathcal{S}$ that have weight at most n is $[x^n]\Phi_{\mathcal{S}}(x)(1-x)^{-1}$.

2. How many ways are there to make change for a dollar, using pennies, nickels, dimes and quarters? (For example, one way is to use 20 pennies, 4 nickels, 1 dime, and 2 quarters.)

3. Let $\mathcal{W}$ be the set of lattice walks in the plane with the following properties:

   - The starting point is $(0,0)$;
   - Each step is a unit step in one of four directions: N, E, S or W;
   - A step is never immediately followed by a step in the opposite direction;
   - Every point of the walk is on or above the x-axis;
   - The final point is on the x-axis;

   Define the weight of a walk to be the number of steps. The goal of this question is to compute the generating function for $\mathcal{W}$.

   Let $\mathcal{X} \subset \mathcal{W}$ be the subset of walks that start with a N step. Let $\mathcal{Y} \subset \mathcal{W}$ be the subset of walks that do not have a N step. Let ϵ denote the empty walk (with 0 steps).

   (a) Show that $\Phi_{\mathcal{Y}}(x) = 1 + 2x(1-x)^{-1}$.

   (b) Find a bijection
   $$\mathcal{Y} \times \mathcal{X} \to \mathcal{W} \setminus \mathcal{Y}\,.$$

   (c) Show that $\Phi_{\mathcal{Y}}(x)\Phi_{\mathcal{X}}(x) = \Phi_{\mathcal{W}}(x) - \Phi_{\mathcal{Y}}(x)$.

   (d) Find a bijection
   $$\mathcal{Y} \times (\mathcal{W} \setminus \{\epsilon\}) \times (\mathcal{W} \setminus \mathcal{X}) \to \mathcal{W} \setminus \mathcal{Y}\,.$$

   (e) Show that $x^2\Phi_{\mathcal{Y}}(x)(\Phi_{\mathcal{W}}(x) - 1)(\Phi_{\mathcal{W}}(x) - \Phi_{\mathcal{X}}(x)) = \Phi_{\mathcal{W}}(x) - \Phi_{\mathcal{Y}}(x)$.

   (f) Show that
   $$\Phi_{\mathcal{W}}(x) = (1+x)\left(\frac{(1-x)^2 - \sqrt{1-4x+6x^2-12x^3+9x^4}}{4x^3}\right)\,.$$

   Detailed proofs are not required for parts (b) and (d), but please include a brief explanation.

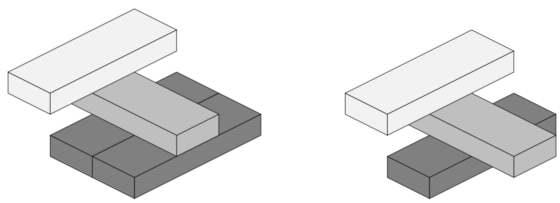
4. A partition $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, is called a *strict partition* if $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$. Prove that the number of strict partitions of n is
   $$[x^n]\prod_{i=1}^{m}(1+x^i)\,, \quad \text{where } m \geq n\,.$$

   Deduce that this number is
   $$[x^n]\prod_{i=1}^{\infty}(1+x^i)\,.$$

5. In the game Jenga, players take turns removing wooden blocks from a tower and placing them on top. At the start of the game, the tower consists of 18 rows of three adjacent blocks. Each block is 3 times as long as it is wide, so three blocks together make a square shape. The rows alternate in orientation, so that the long side of one row is perpendicular to the row below it, and this must remain true when a new row is formed at the top. The game ends when the cat jumps up on the table, knocking over the tower.

   The following are necessary and sufficient conditions for a Jenga tower to be stable (human error and acts of feline notwithstanding).

   - If a row other than the top row has only one block, then that block must be in the centre of the row.
   - If the tower has at least three rows, the top three rows can't look like either of the pictures below (or any rotations thereof):



   A Jenga tower can be represented as a sequence of subsets of $\{1, 2, 3\}$. The sequence specifies where the blocks are in each row of the tower, listed from bottom row to top row. For example, the two unstable towers shown above are $(23, 2, 1)$ and $(2, 2, 1)$ respectively. Two towers are the considered same iff they are represented by the same sequence.

   (a) Let $\mathcal{T}$ be the set of stable Jenga towers with exactly three rows. Determine the generating function for $\mathcal{T}$, if the weight of a tower is the number of blocks.

   (b) How many different stable Jenga towers are possible, using all 54 blocks? [*Hint:* The top 3 rows must form a tower in $\mathcal{T}$.]

   (c) Now suppose we consider a Jenga tower and its rotation by 180° around a vertical axis to be the same object. Under this new interpretation, what is the answer to part (b)?