Math 249, Winter 2013
Assignment 2

Due Wednesday, January 30, in class.

1. (a) If $0 \le a \le b$, prove that the number of lattice paths from $(0,0)$ to (a,b) that never step below the line $y = x$ is $\binom{a+b}{a} - \binom{a+b}{a-1}$. Please provide all relevant details, including those that I did not write down when doing a similar example in class.

   (b) For $n \in \mathcal{N}$, consider lattice paths of length $2n$ that start at $(0,0)$ and have no other points on the line $y = x$. Prove that the number of these paths is $\binom{2n}{n}$.

2. (a) Show that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n \ge 0} \binom{2n}{n} x^n .$$

   (b) Deduce that for all $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} \binom{2j}{j} \binom{2n-2j}{n-j} = 4^n .$$

   (c) Find a combinatorial proof for part (b). [*Hint:* Use 1(b).]

3. A *permutation* of $[n]$ is a bijection $\pi : [n] \to [n]$. The set of all permutations of $[n]$ is denoted S_n. If $\pi \in S_n$, we denote π by listing its values as a finite sequence: $\pi(1), \pi(2), \ldots, \pi(n)$. For example, $\pi = 213$ is the permutation of $[3]$ in which $\pi(1) = 2$, $\pi(2) = 1$, and $\pi(3) = 3$.

   A permutation $\pi \in S_n$ is said to contain a *321-pattern* if there exist $i, j, k \in [n]$ with $i < j < k$ and $\pi(i) > \pi(j) > \pi(k)$. For example, the permutation 321 contains a 321-pattern. If π does not contain a 321-pattern, we say π is *321-avoiding.*

   Describe a bijection between the set of 321-avoiding permuations of $[n]$ and the set of Dyck paths of length $2n$. Illustrate your bijection with a decent example. (You don't need to prove that your bijection is correct.)