Math 249, Winter 2013
Assignment 1

Due Wednesday, January 23, in class.

1. Let X be a finite, non-empty set. Prove that the number of odd subsets X is equal to the number of even subsets of X.

   (An odd subset of X is a subset with an odd number of elements. You can probably guess what an even subset is.)

2. By starting with the binomial theorem and differentiating both sides, prove the following identities for $n \in \mathbb{N}$.

   (a) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

   (b) $\displaystyle\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$

3. Let $A(x) = (1 + x - x^2 - x^3)^{-1}$. Then $A(x)$ has a power series expansion:

$$A(x) = \sum_{n\geq 0} a_n x^n \,.$$

   (a) The coefficients $a_0, a_1, a_2, \ldots$ satisfy a linear recurrence relation. Determine this recurrence relation and its initial conditions.

   (b) Find an explicit formula for a_n, $n \geq 0$.

4. Let $F(n, k)$ denote the number of surjective functions with domain $[n]$ and codomain $[k]$.

   (a) Give a combinatorial proof of the following identity:

$$\sum_{k=0}^{m} \binom{m}{k} F(n,k) 2^{m-k} = \sum_{j=0}^{m} \binom{m}{j} j^n \qquad \text{for all } m, n \in \mathbb{N}\,.$$

   (b) Adapt the argument from part (a) to prove

$$F(n,m) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^n \,.$$

   (Hint: If you did part (a) in a non-roundabout way, the 2^{m-k} factor counts the number of subsets of some $(m - k)$-element set. Consider what happens if instead of treating all of these subsets the same, we "count" the odd-subsets with factor of -1.)