Course Project

C&O 739: Schubert calculus (Winter 2009)

Your assignment is to write an expository paper on a research topic related to the course material, and give a short presentation at the end of term. You may choose from one of the suggested projects below, or propose your own. Regardless, your choice needs to be cleared with me before you begin.

As a rough guide, your paper should be approximately 8–12 pages in length; presentations should be limited to 20–30 minutes.

1. A. Zelevinsky, *A generalization of the Littlewood-Richardson rule and the Robinson-Schensted-Knuth correspondence*, J. Algebra **69** (1981), no. 1, 82–94.

   This paper introduces a combinatorial object called a *picture*, and uses this notion to give a generalization of both the Littlewood-Richardson rule and the Robinson-Schensted-Knuth correspondence (described in Fulton, and many other combinatorics texts).

2. R. King, C. Tollu and F. Toumazet, *Factorisation of Littlewood-Richardson coefficients*, J. Comb. Th. Ser. A., published online (2008).

   In this paper, the authors prove a conjecture of theirs: under certain circumstances, a Littlewood-Richardson number can be written as a product of two Littewood-Richardson numbers, in an interesting and predictable way.

3. A. Buch, *The Saturation conjecture, (after A. Knutson and T. Tao)*, with an appendix by W. Fulton, Enseign. Math. (2) **46** (2000), 43–60

   Let $N > 0$ be an integer. The Saturation theorem states that if $c^{N\lambda}_{N\mu,N\nu} \neq 0$ then $c^{\lambda}_{\mu,\nu} \neq 0$. (Exercise: Prove the converse!) This is an expository paper, explaining Knutson and Tao's proof of the Saturation theorem.

4. A. Knutson and T. Tao, *Puzzles and Equivariant cohomology of Grassmanians.*

   Equivariant cohomology is a generalization of ordinary cohomology, which has certain properties that often makes it easier to work with. The authors explain some of the basic facts about equivariant cohomology, and give a puzzle rule for computing products of Schubert classes in the equivariant cohomology ring of a Grassmannian.

5. H. Thomas and A. Yong, *A combinatorial rule for (co)minuscule Schubert calculus* and *Cominuscule tableau combinatorics*, preprints.

   Minuscule and cominiscule flag varieties are generalizations of flag varieties that are most like Grassmannians. One example is the *maximal orthogonal Grassmannian*, the set of all n-dimensional subspaces $V \subset \mathbb{C}^{2n}$ such that $\sum_{i=1}^{2n} x_i y_i = 0$ for all $x, y \in V$. These papers prove a generalization of the Littlewood-Richardson to all of these varieties.

6. H. Tamvakis, *The connection between representation theory and Schubert calculus.* Enseign. Math. **50** (2004), 267–286.

   The Littlewood-Richardson numbers appear in both representation theory of GL_n and Schubert calculus of the Grassmannian. This expository paper offers a partial, more geometric explanation for this phenomenon.

7. A. Buch, A. Kresch and H. Tamvakis, *Gromov-Witten Invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), 901–915. (See also: H. Tamvakis, *Gromov-Witten invariants and quantum cohomology of Grassmannians.*)

   Quantum cohomology is usually much harder to compute/define than ordinary cohomology—the definition is quite technical in general. However, for flag varieties, many of this technical difficulties don't arise and the definition is much more accessible. This paper gives a very nice result relating the quantum cohomology of a Grassmannian to the ordinary cohomology of two-step flag variety.

8. S. Billey and R. Vakil, *Intersections of Schubert varieties and other permutation array schemes*, IMA Volumes in Mathematics and its Applications, Vol. 146: Algorithms in Algebraic Geometry, (2007) 21–54.

   Finding the points of intersection of Schubert varieties using algebra, amounts to solving a system of equations. This paper discusses some methods for simplifying these equations dramatically, to make the solving process significantly more efficient.

9. V. Lakshmibai and B. Sandhya *Criterion for smoothness of Schubert varieties in* SL_n/B. Proc. Indian Acad. Sci. Math. Sci. **100** (1990), no. 1, 45–52.

   Not every projective variety is a complex manifold. This paper describes answers the question: when is a Schubert variety smooth?

10. B. Kostant, *Lie algebra cohomology and generalized Schubert cells*, Ann. Math, **77** (1963), no. 2, 72–144.

    Lie algebra cohomology gives a completely different way to think about the cohomology ring of a flag variety. In this model, the cohomology ring becomes a subring of the exterior algebra of a certain vector space. Kostant defines *harmonic forms* which are the Schubert classes in this model. A background in differential geometry and Lie theory is needed to read this paper.