

## C&O 739, Winter 2009 Assignment 3

*Due Thursday, March 26, in class.*

1. Let  $G_\bullet \in \mathcal{F}\ell(m)$ . Show that  $F_\bullet \in \Omega_w(G_\bullet)$  if and only if  $G_\bullet \in X_{w^{-1}w_0}$ . (Here and throughout this assignment,  $F_\bullet$  denotes the standard flag in  $\mathcal{F}\ell(m)$ .)

2. Let  $\mathbf{d} = (0 = d_0 < d_1 < \cdots < d_k < d_{k+1} = m)$ . The goal of this exercise is to compute the cohomology ring of the partial-flag variety  $\mathcal{F}\ell_{\mathbf{d}}(m)$ . Let  $\pi : \mathcal{F}\ell(m) \rightarrow \mathcal{F}\ell_{\mathbf{d}}(m)$  be the natural projection.

(a) Show that  $\mathcal{F}\ell(m)$  can be described as (the total space of) a tower of projective-space bundles over  $\mathcal{F}\ell_{\mathbf{d}}(m)$ . Deduce that the ring homomorphism  $\pi^* : H^*(\mathcal{F}\ell_{\mathbf{d}}(m)) \rightarrow H^*(\mathcal{F}\ell(m))$  is injective.

(b) Let

$$W^{\mathbf{d}} = \{w \in \mathcal{S}_m \mid w(i) < w(i+1) \text{ for } i \notin \mathbf{d}\}.$$

Show that every  $T$ -fixed point on  $\mathcal{F}\ell_{\mathbf{d}}(m)$  is of the form  $\pi(wF_\bullet)$ , for a unique  $w \in W^{\mathbf{d}}$ .

(c) Define the Schubert cell  $\Omega_{w,\mathbf{d}}^\circ \subset \mathcal{F}\ell_{\mathbf{d}}(m)$  to be the  $(B_-)$ -orbit of  $\pi(wF_\bullet)$ . Show that every point in  $\mathcal{F}\ell_{\mathbf{d}}(m)$  is in a unique Schubert cell.

(d) The Schubert variety  $\Omega_{w,\mathbf{d}}$  is defined to be the closure of  $\Omega_{w,\mathbf{d}}^\circ$ . Show that  $\pi^{-1}(\Omega_{w,\mathbf{d}}) = \Omega_w$ , the Schubert variety in  $\mathcal{F}\ell(m)$ . (Note:  $\Omega_w^\circ \subset \pi^{-1}(\Omega_{w,\mathbf{d}}^\circ)$ , but they are not equal.)

Since  $\pi$  is a bundle, it follows that  $\pi^*([\Omega_{w,\mathbf{d}}]) = \sigma_w$ . It is also a (not-so-hard) fact that each Schubert cell is isomorphic to  $\mathbb{C}^{\dim \mathcal{F}\ell_{\mathbf{d}}(m) - \ell(w)}$ . By the usual arguments, it follows that the Schubert classes  $\sigma_w$ ,  $w \in W^{\mathbf{d}}$  generate  $H^*(\mathcal{F}\ell_{\mathbf{d}}(m))$ .

(e) From part (a), we can identify  $H^*(\mathcal{F}\ell_{\mathbf{d}}(m))$  with a subring of  $H^*(\mathcal{F}\ell(m)) = R(m)$ . Let  $R_{\mathbf{d}}(m)$  be the subring of  $R(m)$  generated by all  $f$  such that  $r_i(f) = f$  for all  $i \notin \mathbf{d}$ . Show that  $\sigma_w(x_1, \dots, x_m) \in R_{\mathbf{d}}(m)$ , for  $w \in W^{\mathbf{d}}$ .

(f) Let  $U_\bullet$  be the tautological partial-flag bundle on  $\mathcal{F}\ell_{\mathbf{d}}(m)$  ( $U_i$  is a rank- $d_i$  vector bundle). Show that the Chern class  $c_j(U_i/U_{i-1}) \in H^*(\mathcal{F}\ell_{\mathbf{d}}(m))$  equals  $e_j(x_{d_{i-1}+1}, \dots, x_{d_i})$ , where  $e_j$  is the elementary symmetric function.

Since these elementary symmetric functions generate  $R_{\mathbf{d}}(m)$ , we deduce that  $H^*(\mathcal{F}\ell_{\mathbf{d}}(m)) = R_{\mathbf{d}}(m)$ .

3. This problem is an application of Problem 2 to the Grassmannian.

(a) Let  $\pi_d : \mathcal{F}\ell(m) \rightarrow \text{Gr}_d(m)$ . For each  $\lambda \in \text{Part}_{d,m}$ , determine the permutation  $w(\lambda) \in \mathcal{S}$  such that  $\pi_d^*(\sigma_\lambda) = \sigma_{w(\lambda)}$ .

(b) Let  $e_i(x_1, \dots, x_d)$  be the elementary symmetric function for  $i \leq d < m$ . Show explicitly that there is a unique permutation  $w \in \mathcal{S}_m$  such that  $\partial_w e_i(x_1, \dots, x_d) = 1$ . Verify that this permutation is  $w(1^i)$ , where  $1^i$  is the partition with a single column of length  $i$ . Deduce that the Schubert polynomial  $s_{w(1^i)}(x_1, \dots, x_m)$  equals the Schur polynomial  $s_{1^i}(x_1, \dots, x_d)$  in  $d$  variables.

(c) Use the previous results to prove that the Schur polynomial  $s_\lambda(x_1, \dots, x_d)$  equals the Schubert polynomial  $s_{w(\lambda)}(x_1, \dots, x_m)$ , for any  $\lambda$  with at most  $d$  parts. (Hint: This should be fairly easy. There is no need to proceed along the lines in Fulton's book.)

4. If  $w \in \mathcal{S}_m$  is a permutation such that

$$w(1) > w(2) > \dots > w(d) \quad \text{and} \quad w(d) < w(d+1) < \dots < w(m),$$

show that  $s_w(x_1, \dots, x_m) = x_1^{w(1)-1} x_2^{w(2)-1} \dots x_d^{w(d)-1}$ .

5. Show that the following sets generate the ring  $R_{\mathbf{d}}(m)$ , defined in Problem 2(e):

(i) The set of elementary symmetric functions

$$\{e_j(x_1, \dots, x_{d_i}) \mid i = 1, \dots, k, j = 1, \dots, d_i - d_{i-1}\}.$$

(ii) The set of complete symmetric functions

$$\{h_j(x_1, \dots, x_{d_i}) \mid i = 1, \dots, k, j = 1, \dots, d_{i+1} - d_i\}.$$

Note the connection to Problem 3: the  $h_j$ 's in (ii) are Schur polynomials representing special Schubert classes on the Grassmannian. (Hint: (ii) is a pain to do directly. Use duality to deduce (ii) from (i).)